A Functorial Approach to the C^*-algebras of a Graph

Jack Spielberg
Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804
jss@math.la.asu.edu

Abstract.

A functor from the category of directed trees with inclusions to the category of commutative C^*-algebras with injective $*$-homomorphisms is constructed. This is used to define a functor from the category of directed graphs with inclusions to the category of C^*-algebras with injective $*$-homomorphisms. The resulting C^*-algebras are identified as Toeplitz graph algebras. Graph algebras are proved to have inductive limit decompositions over any family of subgraphs with union equal to the whole graph. The construction is used to prove various structural properties of graph algebras.

Introduction.

Since the paper of Cuntz and Krieger in 1976, much work has gone into elucidating the brief remarks made there regarding the case of infinite $0-1$ matrices. While perhaps the most far-reaching solution put forward has been a direct generalization to infinite $0-1$ matrices (**[9]**), most of the papers on the subject generalize to a class of infinite directed graphs. (In fact, this is the direction indicated in **[7]**.) In this paper we propose a new method of accomplishing this generalization to infinite graphs, based on a construction of a boundary for an arbitrary directed tree. For a locally finite tree, (or more generally, a directed tree in which each vertex emits finitely many edges), the obvious space of (directed) ends serves as a boundary. Without the finiteness condition, however, the space of ends is not locally compact. This difficulty has been solved in certain cases by ad-hoc methods (e.g., **[21,13,23]**). More recently these difficulties have been overcome more generally, though again by ad-hoc methods (**[8,20]**). Our approach is to give a natural solution in the form of a functor from the category of directed trees (with inclusion) to the category of commutative C^*-algebras (with injective $*$-homomorphisms). In the case of a locally finite tree, this functor yields the algebra of continuous functions vanishing at infinity on the space of ends. Given a directed graph, the (combinatorial) universal cover is a bundle of directed trees, on which acts the (combinatorial) fundamental groupoid. Applying our boundary functor, we may let the groupoid act on the resulting C^*-algebra. We prove that this yields the same graph algebra (up to strong Morita equivalence) as has been defined by others. The chief virtue of our approach seems to us to be its naturality. We obtain a morphism of graph algebras from an inclusion of directed graphs. This leads us to introduce a family of Toeplitz graph algebras associated to a directed graph. We immediately obtain an inductive limit decomposition of a graph algebra over any directed family of subgraphs. We use this theorem in **[23]** to establish semiprojectivity for classifiable simple purely infinite C^*-algebras having finitely generated K-theory and torsion-free K_1. Moreover, we mention that all of the structure of the graph algebras is obtained without any restrictions on the size or structure of the graph. (A different method for obtaining the direct limit decomposition over the finite subgraphs is given in **[20]**.)

The outline of the paper is as follows. We finish the introduction with the basic notation we will use. We find it convenient to separate the *graphical* properties of an edge from the *groupoid* properties. Thus we adopt the terminology of *origin* and *terminus*, as in **[22]**, leaving *source* and *range* exclusively as groupoid

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terms. In section 1 we introduce our boundary functor for directed trees. We include a classification of the open subsets of the boundary, as this is needed to classify the ideals in the graph algebras. In section 2 we construct the groupoids and C^*-algebras obtained from a directed graph by following the above approach. We prove the standard presentation by generators and relations, and define the fundamental cocycle. This allows us to exhibit the AF core as the C^*-algebra of an equivalence relation, and prove nuclearity for all graph algebras. We also prove the inductive limit decomposition described above. In section 3 we prove the basic structure theorems regarding simplicity and pure infiniteness. These properties follow quite easily from the theorems already established.

The idea of using a Boolean ring of subsets of the vertices of a(n undirected) tree in order to define its boundary was discovered jointly with Marcelo Laca, when the author was visiting the Mathematics Department at the University of Newcastle. He wishes to thank the members of the Department, and Marcelo Laca and Iain Raeburn in particular, for their wonderful generosity and hospitality during that stay.

Following **[22]** we let a *graph*, E, consist of a set E^0 of *vertices*, a set E^1 of *edges*, maps $o,\, t : E^1 \to E^0$ (*origin* and *terminus*), and a map $e \mapsto \overline{e}$ on E^1 (*reversal*), satisfying:

$$\overline{\overline{e}} = e$$
$$\overline{e} \neq e$$
$$o(\overline{e}) = t(e).$$

A *directed graph* consists of a graph E together with a subset $E^1_+ \subseteq E^1$ containing exactly one edge from each pair $\{e, \overline{e}\}$. (When drawing E it is usual to give only one segment to represent $\{e, \overline{e}\}$. If E is directed, the segment representing $\{e, \overline{e}\}$ is given an arrowhead pointing toward $t(|e|)$, where $|e|$ is the element of $\{e, \overline{e}\} \cap E^1_+$.) A *path* in E is a (finite or infinite) sequence of edges $e_1 e_2 \cdots$ such that $t(e_{i-1}) = o(e_i)$ and $e_i \neq \overline{e_{i-1}}$ for all $i > 1$. We write $o(e_1 e_2 \cdots) = o(e_1)$, $t(e_1 \cdots e_n) = t(e_n)$, and $\overline{e_1 \cdots e_n} = \overline{e_n} \cdots \overline{e_1}$. If p is a path in E we let $\ell(p)$ denote the *length* of p $(0 \leq \ell(p) \leq \infty)$. If E is a directed graph we call a path $e_1 e_2 \cdots$ *directed* if $e_i \in E^1_+$ for all i. (A path of length zero consists of a single vertex, and is considered to be directed.) We let E^n (respectively E^n_+) denote the paths (respectively directed paths) of length n, for $0 \leq n \leq \infty$, $E^* = \bigcup_{0 \leq n < \infty} E^n$, $E^*_+ = \bigcup_{0 \leq n < \infty} E^n_+$, and $E^{**} = E^* \cup E^\infty$, $E^{**}_+ = E^*_+ \cup E^\infty_+$. The graph E is a *tree* if for all $u, v \in E^0$ there exists a unique path $p \in E^*$ with $o(p) = u$ and $t(p) = v$. A graph is a *forest* if it is a disjoint union of trees. More precisely, the graph E is a forest if for all $u, v \in E^0$ there exists at most one path $p \in E^*$ with $o(p) = u$ and $t(p) = v$.

Let E be a directed graph. For $v \in E^0$ let

$$\begin{aligned} V(v) \equiv V_E(v) &= \big\{ t(p) \bigm| p \in E^*_+,\ o(p) = v \big\}, \\ \Delta_1(v) \equiv \Delta_{1,E}(v) &= \big\{ e \in E^1_+ \bigm| o(e) = v \big\} \\ \Omega \equiv \Omega(E) &= \big\{ u \in E^0 \bigm| \Delta_1(u) = \emptyset \big\} \\ \Sigma \equiv \Sigma(E) &= \big\{ u \in E^0 \bigm| 0 < \#\Delta_1(u) < \infty \big\}, \end{aligned}$$

and for $F \subseteq \Delta_1(v)$ a finite subset,

$$V(v; F) \equiv V_E(v; F) = \{v\} \cup \bigcup_{e \in \Delta_1(v) \setminus F} V\big(t(e)\big).$$

Thus $V(v)$ is the set of vertices that can be found by following a finite directed path from v, $V(v; F)$ is the set of vertices that can be found by following a directed path from v that does not begin with an edge from

the finite set F, and Ω is the set of vertices often termed *sinks.* If $e \in \Delta_1(v)$, we will also write $V(v;e)$ for $V(v;\{e\})$. We will also use the following notations for paths, respectively directed paths, ending at a sink:

$$\begin{aligned}\Omega^* &= t^{-1}(\Omega) \cap E^* \\ \Omega^*_+ &= \Omega^* \cap E^*_+.\end{aligned}$$

1. The Boundary of a Directed Tree.

Remark 1.1. Suppose that T is a directed tree, and $v \in T^0$. If e, $f \in \Delta_1(v)$ are distinct, then $V\big(t(e)\big) \cap V\big(t(f)\big) = \emptyset$.

It follows that for a directed tree T, we may also write

$$V_T(v;F) = V_T(v) \setminus \bigcup_{e \in F} V_T\big(t(e)\big).$$

For T a directed tree, we define

$$\begin{aligned}\mathcal{E} \equiv \mathcal{E}(T) &= \big\{V(v;F) \bigm| v \in T^0,\ F \subseteq \Delta_1(v) \text{ finite}\big\} \cup \big\{\emptyset\big\}, \\ \mathcal{A} \equiv \mathcal{A}(T) &= \big\{ \cup_{i=1}^k B_i \bigm| B_1, \ldots, B_k \in \mathcal{E} \text{ are disjoint},\ k \geq 1\big\}.\end{aligned}$$

Lemma 1.2. *If B, $C \in \mathcal{E}$, then $B \cap C$, $B \setminus C \in \mathcal{A}$.*

Proof. Let $B = V(u;F)$, $C = V(v;G)$, where $F = \{f_1, \ldots, f_k\}$, $G = \{g_1, \ldots, g_\ell\}$. Suppose first that $u = v$. Then $B \cap C = V(u; F \cup G)$ and $B \setminus C = \bigcup_{e \in G \setminus F} V\big(t(e)\big)$; both are in $\mathcal{A}$ (the second by Remark 1.1).

Now suppose that $u \neq v$. We distinguish three cases.

case (i): $v \in V(u)$. Let $e_1 \cdots e_n$ be the (unique directed) path from u to v. If $e_1 \in F$, then $V\big(t(e_1)\big) \cap B = \emptyset$. Since $C \subseteq V\big(t(e_1)\big)$, it follows that $B \cap C = \emptyset$, $B \setminus C = B$, and $C \setminus B = C$. If $e_1 \notin F$, then $C \subseteq V(v) \subseteq B$, so $B \cap C = C$. Thus $C \setminus B = \emptyset$, and

$$B \setminus C = V\big(u; F \cup \{e_1\}\big) \cup V\big(t(e_1); e_2\big) \cup \cdots \cup V\big(t(e_{n-1}); e_n\big) \cup \bigcup_{e \in G} V\big(t(e)\big),$$

a disjoint union of sets from $\mathcal{E}$.

case (ii): $u \in V(v)$. This case is analogous to case (i).

case (iii): $u \notin V(v)$ and $v \notin V(u)$. If $V(u) \cap V(v) = \emptyset$, then $B \cap C = \emptyset$ and $B \setminus C = B$. Otherwise, the path from u to v has the form $p\overline{q}$, where p, $q \in E^*_+$. Then $B \cap C = V\big(t(p)\big)$, and $B \setminus C = B \setminus V\big(t(p)\big)$, which falls under case (i). ∎

Lemma 1.3. *$\mathcal{A}$ is a (Boolean) ring of sets.*

Proof. It is a standard fact that if $\mathcal{E}$ is a collection of sets containing $\emptyset$, $\mathcal{A}$ is the collection of finite disjoint unions of sets from $\mathcal{E}$, and if for any B, $C \in \mathcal{E}$ we have $B \cap C$, $B \setminus C \in \mathcal{A}$, then $\mathcal{A}$ is a ring of sets. ∎

Before continuing, we need some more properties of sets in the ring $\mathcal{A}(T)$. In the following lemma, the notation $V(x;y)$ will mean that $x \in T^0$ and that y is a finite subset of $\Delta_1(x)$.

Lemma 1.4. *Let T be a directed tree.*

(i) $V(u;F) \subseteq V(v;G)$ if and only if one of the following two conditions holds:

(a) $u = v$ and $F \supseteq G$, or

(b) $u \neq v$, and if $p = e_1 e_2 \cdots \in T^*$ is the unique path with $o(p) = v$ and $t(p) = u$, then $p \in T^*_+$ and $e_1 \notin G$.

(ii) If $V(u;F) \subseteq V(v;G)$, then $v \in V(u;F) \iff u = v$.

(iii) If $V(v;G) = \bigcup_{i=1}^k V(u_i;F_i)$ is a disjoint union, then

(a) $u_i = v$ for some i, say $i = 1$,

(b) $F_1 \supseteq G$,

(c) $\displaystyle\bigcup_{i=2}^k V(u_i;F_i) = \bigcup_{e \in F_1 \setminus G} V\big(t(e)\big)$.

Proof. (i) The *if* is clear. For the *only if*, part (a) is clear. Suppose that $u \neq v$. Then part (b) follows from the fact that $u \in V(v;G)$.

(ii) Let p be as in (i). Then $p \in T^*_+$ by (i). Now $v \in V(u;F) \iff \overline{p} \in T^*_+ \iff u = v$.

(iii) Part (a) follows from (ii). Then part (b) follows from (ia). Finally, for part (c) note that

$$\begin{aligned} \bigcup_2^k V(u_i;F_i) &= V(v;G) \setminus V(v;F_1) \\ &= \bigcup_{e \in F_1 \setminus G} V\big(t(e)\big). \qquad \blacksquare \end{aligned}$$

In the next result we charactize Boolean ring homomorphisms from $\mathcal{A}(T)$.

Lemma 1.5. *Let Θ be a Boolean ring, and for $u \in T^0$ let $\theta_u \in \Theta$ satisfy*

(i) $e \in \Delta_1(v) \implies \theta_{t(e)} \subseteq \theta_v$

(ii) $e_1, e_2 \in \Delta_1(v)$ and $e_1 \neq e_2 \implies \theta_{t(e_1)} \cap \theta_{t(e_2)} = \emptyset$.

Then there exists a unique Boolean ring homomorphism $\mu : \mathcal{A}(T) \to \Theta$ such that $\mu\big(V(u)\big) = \theta_u$, for $u \in T^0$.

Proof. We claim that the assignment $V(u;F) \mapsto \theta_u \setminus \bigcup\{\theta_{t(e)} \mid e \in F\}$ is well-defined. First, if $V(u;F) = V(v;G)$, then by Lemma 1.4 (i) and (ii) we have $u = v$ and $F = G$. Suppose inductively that $\mu(B) = \cup_i \mu(B_i)$ whenever $B, B_i \in \mathcal{E}$ and $B = \cup_i B_i$ is a disjoint union of fewer than n sets, and let $V(v;G) = \cup_{i=1}^n V(u_i;F_i)$ be a disjoint union. By Lemma 1.4, possibly after reordering the $\{u_i\}$, we have

$$\begin{aligned} &u_1 = v \\ &V(v;G) = V(v;F_1) \cup \bigcup_{e \in F_1 \setminus G} V\big(t(e)\big), \text{ a disjoint union,} \\ &V\big(t(e)\big) = \bigcup\big\{V(u_i;F_i) \bigm| u_i \in V\big(t(e)\big)\big\}, \text{ for } e \in F_1 \setminus G. \end{aligned}$$

Since $G \subseteq F_1$, we have

$$\theta_v \setminus \bigcup_{e \in G} \theta_{t(e)} = \big(\theta_v \setminus \bigcup_{e \in F_1} \theta_{t(e)}\big) \cup \bigcup_{f \in F_1 \setminus G} \theta_{t(f)}.$$

For $e \in F_1 \setminus G$, $\#\{i \mid u_i \in V_{T_1}\big(t(e)\big)\} < n$. By the inductive hypothesis, we have for each $e \in F_1 \setminus G$,

$$\theta_{t(e)} = \bigcup_{u_i \in V\big(t(e)\big)} \mu\big(V(u_i; F_i)\big).$$

Thus

$$\begin{aligned}\mu\big(V(v;G)\big) &= \mu\big(V(v;F_1)\big) \cup \bigcup_{e \in F_1 \setminus G} \theta_{t(e)} \\ &= \mu\big(V(v;F_1)\big) \cup \bigcup_{e \in F_1 \setminus G} \bigcup_{u_i \in V\big(t(e)\big)} \mu\big(V(u_i;F_i)\big) \\ &= \bigcup_{i=1}^{n} \mu\big(V(u_i;F_i)\big).\end{aligned}$$

This shows that μ is well-defined on elements of $\mathcal{E}$. It follows that μ is well-defined on $\mathcal{A}$ by $\mu(\cup_i B_i) = \cup_i \mu(B_i)$ for finite disjoint collections $\{B_i\} \subseteq \mathcal{E}$. By following the proof of Lemma 1.2, it is now easy to see that $\mu(B \cap C) = \mu(B) \cap \mu(C)$ and $\mu(B \setminus C) = \mu(B) \setminus \mu(C)$ for any $B, C \in \mathcal{E}$, and hence for $B, C \in \mathcal{A}$. Thus μ is a homomorphism of Boolean rings. Uniqueness follows from the fact that $\mathcal{A}$ is generated by $\{V(u) \mid u \in T^0\}$. ∎

Remark 1.6. With μ as in Lemma 1.5, $\ker(\mu)$ is the set of finite unions of sets of the form $V(u;F)$ for which $\theta_u = \bigcup_{e \in F} \theta_{t(e)}$.

We now let $A \equiv A(T) \subseteq \ell^\infty(T^0)$ be the closed linear span of $\big\{\chi_B \;\big|\; B \in \mathcal{A}\big\}$. Then A is a commutative C^*-algebra, and $\mathcal{A}$ can be identified with the Boolean ring of projections in A. We note that there is a bijective correspondence between the $*$-representations of A into a C^*-algebra L, and the ring homomorphisms of $\mathcal{A}$ into the ring of projections in a commutative C^*-subalgebra of L. (The correspondence is obtained as follows: if $\phi : A \to L$ is a $*$-homomorphism, then define $\mu : \mathcal{A} \to Proj\big(\phi(A)\big)$ by $\mu(B) = \phi(\chi_B)$. Moreover, $\ker \phi = \overline{\text{span}}\{\chi_B \mid \mu(B) = 0\}$.) We will identify $\widehat{A}$ with the set of $\mathcal{A}$-ultrafilters: for $\omega \in \widehat{A}$ we have the $\mathcal{A}$-ultrafilter $\{B \in \mathcal{A} \mid \langle \omega, \chi_B \rangle = 1\}$. (It is elementary to show that this coincides with the Stone space of $\mathcal{A}$ ([**12**], 5.S).) Of course the fixed $\mathcal{A}$-ultrafilters correspond to the elements of T^0: for $v \in T^0$, $\mathcal{U}_v = \{B \in \mathcal{A} \mid v \in B\}$. We will characterize the free $\mathcal{A}$-ultrafilters.

For $p = e_1 e_2 \cdots \in T_+^\infty$ let $\mathcal{U}_p = \big\{B \in \mathcal{A} \;\big|\; V\big(o(e_j)\big) \subseteq B \text{ for some } j\big\}$. It is clear that $\mathcal{U}_p$ is an $\mathcal{A}$-filter.

Lemma 1.7. *$\mathcal{U}_p$ is an $\mathcal{A}$-ultrafilter.*

Proof. Let $B \in \mathcal{A}$, $B \neq \emptyset$, $B \notin \mathcal{U}_p$. Write $B = \cup_1^k B_i$ with $\{B_i\} \subseteq \mathcal{E}$ disjoint. Then $B_i \notin \mathcal{U}_p$ for each i. Let $B_i = V(v_i; F_i)$, and let $q_i \in T^*$ be the geodesic from v_i to p (cf. [**22**], 6.3 Lemma 9). Since $B_i \notin \mathcal{U}_p$, then either $q_i \notin T_+^*$ or the first edge of q_i is in F_i. In either case, it follows that if $t(q_i) = o(e_{n_i})$, then $B_i \cap V\big(t(e_{n_i})\big) = \emptyset$. Let $n = \max(n_1, \ldots, n_k)$. Then $V\big(o(e_n)\big) \in \mathcal{U}_p$ and $B \cap V\big(o(e_n)\big) = \emptyset$. Hence $\mathcal{U}_p$ is a maximal $\mathcal{A}$-filter. ∎

Definition 1.8. (cf. [**15**], section 4) If $p = e_1 e_2 \cdots$ and $q = f_1 f_2 \cdots \in T_+^\infty$ we say that $p \sim q$ if there exist j and k so that $e_{j+r} = f_{k+r}$ for $r \geq 0$.

It is clear that $p \sim q$ if and only if $\mathcal{U}_p = \mathcal{U}_q$. We let $\mathcal{U}_{[p]}$ denote the common $\mathcal{A}$-ultrafilter corresponding to the infinite paths in $[p] \in T_+^\infty / \sim$.

Lemma 1.9. $\widehat{A} = T^0 \cup (T_+^\infty / \sim)$.

Proof. Let $\mathcal{U}$ be a free $\mathcal{A}$-ultrafilter. Let $V(v) \in \mathcal{U}$, for some $v \in T^0$. We claim that there is $e \in E_+^1$ with $v = o(e)$ and $V\big(t(e)\big) \in \mathcal{U}$. For suppose not. Then for any finite set $F \subseteq \Delta_1(v)$, $V(v; F) \in \mathcal{U}$ (we have used the fact that if a finite union of disjoint sets is contained in an $\mathcal{A}$-ultrafilter, then one of the sets must be in the $\mathcal{A}$-ultrafilter). But then $B \in \mathcal{U}$ whenever $B \in \mathcal{A}$ and $v \in B$. Thus $\mathcal{U} \supseteq \mathcal{U}_v$, and hence $\mathcal{U} = \mathcal{U}_v$ is a not a free $\mathcal{A}$-ultrafilter. This establishes the claim. Since $\mathcal{U} \neq \emptyset$ there exists $v \in T_0$ with $V(v) \in \mathcal{U}$. Then we may use the claim to find $p = e_1 e_2 \cdots \in T_+^\infty$ such that $V\big(t(e_j)\big) \in \mathcal{U}$ for all j. Then $\mathcal{U} \supseteq \mathcal{U}_{[p]}$. Since $\mathcal{U}_{[p]}$ is an $\mathcal{A}$-ultrafilter, $\mathcal{U} = \mathcal{U}_{[p]}$. ∎

Remark 1.10. We will extend the definition of the equivalence $\sim$ from T_+^∞ to $T_+^\infty \cup T_+^*$: for p, $q \in T_+^*$ we say that $p \sim q$ if $t(p) = t(q)$. For example, while it is true that $\Omega_+^* / \sim \equiv \Omega$, it will prove useful to have the classes represented by paths.

We next define the boundary of a directed tree. Our definition coincides with the usual space of ends in the case where $\Delta_1(u)$ is finite for all vertices u (that is, in the case of a *row-finite* tree).

Definition 1.11. The *boundary* of the directed tree T is the closure in $\widehat{A}$ of the set of equivalence classes of infinite directed paths and finite directed paths ending in Ω:

$$\partial T = \overline{(T_+^\infty \cup \Omega_+^*) / \sim}\,.$$

Remark 1.12. For $B \in \mathcal{A}$, we will let $[B]$ denote the support of χ_B in $\widehat{A}$ (that is, the set of $\mathcal{A}$-ultrafilters containing B). We will let $[B]_\partial = [B] \cap \partial T$.

Remark 1.13. For $v \in T^0$, a neighborhood base at v in $\widehat{A}$ is given by $\big\{[V(v; F)] \bigm| F \subseteq \Delta_1(v) \text{ finite}\big\}$. For $p = e_1 e_2 \cdots \in T_+^\infty$, a neighborhood base at $[p]$ in $\widehat{A}$ is given by $\big\{\big[V\big(o(e_j)\big)\big] \bigm| j = 1, 2, \ldots\big\}$.

Lemma 1.14. *Let $v \in T^0$. Then $v \in \partial T$ iff $\Delta_1(v)$ is infinite or empty.*

Proof. If $\Delta_1(v)$ is empty, then $v \in \Omega \subseteq \partial T$. If $\Delta_1(v)$ is infinite, then for every finite $F \subseteq \Delta_1(v)$, there is $p = e_1 e_2 \cdots \in T_+^\infty \cup \Omega_+^*$ with $e_1 \notin F$. Then $[p] \in \big[V(v; F)\big]$. Hence $v \in \overline{(T_+^\infty \cup \Omega_+^*) / \sim}$. If $\Delta_1(v)$ is nonempty and finite, then $\{v\} = V\big(v; \Delta_1(v)\big)$. Hence $v \notin \partial T$. ∎

Remark 1.15. $\Sigma(T) = A(T)\widehat{\ } \setminus \partial T$, and is a discrete clopen subset of $A(T)\widehat{\ }$.

Definition 1.16. For $S \subseteq \Sigma$ we let

$$\begin{aligned} A(T, S) &= A(T)/C_0(S) \cong C_0\big(\partial T \cup (\Sigma \setminus S)\big) \\ \partial(T, S) &= A(T, S)\widehat{\ } \cong S^c. \end{aligned}$$

Thus $A(T, \emptyset) = A$ and $A(T, \Sigma) = C_0(\partial T)$; $\partial(T, \emptyset) = \widehat{A}$, and $\partial(T, \Sigma) = \partial T$.

Lemma 1.17. *Let $S \subseteq \Sigma(T)$, let $\pi : A(T) \to A(T, S)$ be the quotient map, and let $\theta : \mathcal{A}(T) \to$ Proj $\big(A(T, S)\big)$ be the corresponding Boolean ring homomorphism. Then $\ker \theta$ equals the collection of finite subsets of S.*

Proof. We have $\Sigma(T) \subseteq \mathcal{A}(T)$ via the singleton sets $\big\{V\big(u; \Delta_1(u)\big) \bigm| u \in \Sigma(T)\big\}$. The finite unions of these sets are precisely the elements of $\mathcal{A}(T)$ sent to $\emptyset$ by θ. ∎

Next we discuss the behavior of the algebras $A(T, S)$ under mappings of trees. By a *morphism of directed graphs* we mean a pair of maps from the set of vertices (edges) of the first graph to the set of vertices (edges) of the second graph that intertwine o, t, and $\overline{(\cdot)}$, and map directed edges to directed edges. It is a *monomorphism* if it is injective on the sets of vertices and edges.

Lemma 1.18. *Let $\alpha : T_1 \to T_2$ be a monomorphism of directed trees. Then*

$$V_{T_1}(v;F) \mapsto \alpha_*\big(V_{T_1}(v;F)\big) \equiv V_{T_2}\big(\alpha(v);\alpha(F)\big)$$
$$\cup_i B_i \mapsto \cup_i \alpha_*(B_i) \ (\{B_i\} \subseteq \mathcal{E}(T_1) \text{ finite disjoint})$$

defines a Boolean ring monomorphism $\alpha_* : \mathcal{A}(T_1) \to \mathcal{A}(T_2)$.

Proof. We will check the conditions of Lemma 1.5. Let $e \in \Delta_{1,T_1}(v)$. Then $\alpha(e) \in \Delta_{1,T_2}\big(\alpha(v)\big)$, so $V_{T_2}\big(\alpha\big(t(e)\big)\big) \subseteq V_{T_2}\big(\alpha(v)\big)$. Now let $e_1 \neq e_2 \in \Delta_{1,T_1}(v)$. Since α is one-to-one on edges, $\alpha(e_1) \neq \alpha(e_2) \in \Delta_{1,T_2}\big(\alpha(v)\big)$. Thus, since T_2 is a tree, $V_{T_2}\Big(t\big(\alpha(e_1)\big)\Big) \cap V_{T_2}\Big(t\big(\alpha(e_2)\big)\Big) = \emptyset$. By Lemma 1.5 there is a Boolean ring homomorphism $\alpha_* : \mathcal{A}(T_1) \to \mathcal{A}(T_2)$ such that $\alpha_*\big(V_{T_1}(v)\big) = V_{T_2}\big(\alpha(v)\big)$, for $v \in T_1^0$. Then

$$\begin{aligned}
\alpha_*\big(V_{T_1}(v;F)\big) &= \alpha_*\big(V_{T_1}(v) \setminus \bigcup_{e\in S} V_{T_1}\big(t(e)\big)\big) \\
&= V_{T_2}\big(\alpha(v)\big) \setminus \bigcup_{e\in S} V_{T_2}\big(\alpha\big(t(e)\big)\big) \\
&= V_{T_2}\big(\alpha(v)\big) \setminus \bigcup_{f\in\alpha(S)} V_{T_2}\big(t(f)\big) \\
&= V_{T_2}\big(\alpha(v);\alpha(F)\big).
\end{aligned}$$

Finally α_* is a monomorphism since B nonempty implies $\alpha_*(B)$ nonempty. ∎

Remark 1.19. If $\alpha : T_1 \to T_2$ is a monomorphism of directed trees, then the ring monomorphism $\alpha_* : \mathcal{A}(T_1) \to \mathcal{A}(T_2)$ induces an injective $*$-homomorphism $\alpha_* : A(T_1) \to A(T_2)$.

Lemma 1.20. *Let $\alpha : T_1 \to T_2$ be a monomorphism of directed trees, and let $S_2 \subseteq \Sigma(T_2)$. Let*

$$S_1 = \big\{v \in \Sigma(T_1) \bigm| \alpha(v) \in S_2 \text{ and } \alpha\big(\Delta_{1,T_1}(v)\big) = \Delta_{1,T_2}\big(\alpha(v)\big)\big\}.$$

Then $\ker(\pi_2 \circ \alpha_*) = C_0(S_1)$*, so that the following diagram commutes, has surjective columns (given by restrictions), and has injective rows:*

$$\begin{array}{ccc}
A(T_1) & \xrightarrow{\alpha_*} & A(T_2) \\
\Big\downarrow & & \Big\downarrow \pi_2 \\
A(T_1,S_1) & \longrightarrow & A(T_2,S_2).
\end{array}$$

Proof. Let $\theta : \mathcal{A}(T_1) \to \operatorname{Proj}\big(A(T_2,S_2)\big)$ be the Boolean ring homomorphism associated to $\pi_2 \circ \alpha_*$. Then $V_{T_1}(v;F) \in \ker\theta$ if and only if

$$\theta\big(V_{T_1}(v)\big) = \bigcup_{e\in F} \theta\big(V_{T_1}\big(t(e)\big)\big).$$

Since α_* is a monomorphism, this happens if and only if the characteristic function of $\big[V\big(\alpha(v);\alpha(F)\big)\big]$ is in $\ker\pi_2$. By Lemma 1.17, this happens if and only if $\alpha(v) \in S_2$, $F = \Delta_{1,T_1}(v)$, and $\alpha\big(\Delta_{1,T_1}(v)\big) = \Delta_{1,T_2}\big(\alpha(v)\big)$. ∎

Remark 1.21. If $\alpha : T_1 \to T_2$ and $\beta : T_2 \to T_3$ are monomorphisms of directed trees, then it is clear that $\beta_* \circ \alpha_* = (\beta\circ\alpha)_*$ at the level of Boolean rings, and hence also as $*$-homomorphisms. Let $S_3 \subseteq \Sigma(T_3)$. Applying Lemma 1.20 to β and S_3 yields $S_2 \subseteq \Sigma(T_2)$. Applying Lemma 1.20 to α and S_2 yields $S_1 \subseteq \Sigma(T_1)$. It is easily verified that an application of Lemma 1.20 to $\beta\circ\alpha$ and S_3 would give the same subset S_1 of $\Sigma(T_1)$. Hence the concatenation of commuting squares obtained from Lemma 1.20 gives the commuting square for a composition.

Corollary 1.22. *Let* $\pi : A(T_2) \to C_0(\partial T_2)$ *be the restriction mapping. Then* $\ker(\pi \circ \alpha_*) = C_0(S)$, *where*

$$S = \bigl\{ v \in \Sigma(T_1) \bigm| \alpha\bigl(\Delta_{1,T_1}(v)\bigr) = \Delta_{1,T_2}\bigl(\alpha(v)\bigr) \bigr\}. \qquad \blacksquare$$

We next characterize the open subsets of the boundary of a directed tree. This will be crucial for determining the ideals in graph C^*-algebras later on. We will eventually need the generalizations of these notions to directed graphs. Therefore, we state the definitions now in that generality.

Definition 1.23. Let E be a directed graph. An *invariant* of E is a pair (N, F) consisting of a subset $N \subseteq E^0$ and a family $\{F_u \mid u \in N\}$, where $F_u \subseteq \Delta_1(u)$ is a finite set, satisfying the following conditions.

(i) If $u \in N$ and $\Delta_1(u)$ is finite, then $F_u = \emptyset$.

(ii) Let $u \in N$ and $e \in \Delta_1(u)$. Then

  (a) If $e \notin F_u$, then $t(e) \in N$ and $F_{t(e)} = \emptyset$.

  (b) If $e \in F_u$ and $t(e) \in N$, then $F_{t(e)} \neq \emptyset$.

(iii) Let $u \in E^0$ with $\Delta_1(u)$ finite and nonempty. If $t\bigl(\Delta_1(u)\bigr) \subseteq N$, and $F_{t(e)} = \emptyset$ for all $e \in \Delta_1(u)$, then $u \in N$.

We let $\mathcal{I} \equiv \mathcal{I}(E)$ denote the set of invariants of E. We remark that the notions of *hereditary* and *saturated* subsets of E^0 appearing in other work are being replaced by conditions (i) and (ii)(a), respectively (ii)(b) and (iii), in the context of general directed graphs. We define a partial ordering in $\mathcal{I}$ by $(N_1, F_1) \leq (N_2, F_2)$ if $N_1 \subseteq N_2$ and $F_{1,u} \supseteq F_{2,u}$ for each $u \in N_1$.

Example 1.24. Let $E^0 = \mathbf{Z}$ and let E have infinitely many directed edges from n to $n+1$, for each $n \in E^0$. Then for $n \in \mathbf{Z}$ we may let $N = [n, \infty)$ and $F_k = \emptyset$ for $k \in N$. These are the only elements of $\mathcal{I}(E)$.

We again specialize to directed trees.

Definition 1.25. Let T be a directed tree. Let $\mathcal{J} \equiv \mathcal{J}(T)$ be the set of open subsets of ∂T. We let $\mathcal{J}$ be ordered by inclusion. We define maps $U_\partial : \mathcal{I} \to \mathcal{J}$ and $L : \mathcal{J} \to \mathcal{I}$ as follows.

$$U_\partial(N, F) = \bigcup_{u \in N} \bigl[V(u; F_u)\bigr]_\partial$$

$$L(W) = (N, F) \equiv \bigl(N(W), F(W)\bigr),$$

where

$$\begin{aligned} N = &\bigl\{ u \in T^0 \setminus \Sigma(T) \bigm| \exists F \subseteq \Delta_1(u) \text{ finite with } \bigl[V(u; F)\bigr]_\partial \subseteq W \bigr\} \\ &\cup \bigl\{ u \in T^0 \bigm| \bigl[V(u)\bigr]_\partial \subseteq W \bigr\}, \end{aligned}$$

$$F_u = \bigcap \bigl\{ F \subseteq \Delta_1(u) \bigm| \bigl[V(u; F)\bigr]_\partial \subseteq W \bigr\}, \quad u \in N.$$

Lemma 1.26. *The above definitions make sense, in that* $U_\partial(N, F) \in \mathcal{J}$ *and* $L(W) \in \mathcal{I}$ *whenever* $(N, F) \in \mathcal{I}$ *and* $W \in \mathcal{J}$.

Proof. Left to the reader. $\blacksquare$

Remark 1.27. It is immediate that the maps L and U_∂ are order-preserving.

Theorem 1.28. *The maps L and U_∂ are inverses.*

Proof. We first show that $U_\partial \circ L$ is the identity on $\mathcal{J}$. Let $W \in \mathcal{J}$, and let $L(W) = (N, F)$. Let $u \in N$. Then $\left[V(u; F_u)\right]_\partial$ is one of the defining sets of $U_\partial\big(L(W)\big)$. By the definition of $L(W)$ we have $\left[V(u; F_u)\right]_\partial \subseteq W$. It follows that $U_\partial\big(L(W)\big) \subseteq W$.

For the reverse inclusion, note that since W is open, it is a union of basic open sets. Let $\left[V(u; F)\right]_\partial \subseteq W$ be one, where F is a finite subset of $\Delta_1(u)$. We may assume that F is the smallest subset of $\Delta_1(u)$ for which the containment holds. Now, if $u \in \partial T$, then $u \in N$. It follows that $\left[V(u; F)\right]_\partial \subseteq U_\partial\big(L(W)\big)$. The same argument holds if $u \notin \partial T$ but $u \in N$. Finally, we suppose that $u \notin N$. Then we must have $u \notin \partial T$ and $\left[V(u)\right]_\partial \not\subseteq W$. Then $\Delta_1(u)$ is finite. We have

$$\begin{aligned} W &\supseteq \left[V(u; F)\right]_\partial \\ &= \bigcup_{e \in \Delta_1(u) \setminus F} \left[V\big(t(e)\big)\right]_\partial. \end{aligned}$$

Since $\left[V\big(t(e)\big)\right]_\partial \subseteq W$ for such e, we have $t(e) \in N$. Then $\left[V\big(t(e)\big)\right]_\partial \subseteq U_\partial\big(L(W)\big)$. It now follows that $\left[V(u; F)\right]_\partial \subseteq U_\partial\big(L(W)\big)$. Therefore $W \subseteq U_\partial\big(L(W)\big)$.

We next show that $L \circ U_\partial$ is the identity on $\mathcal{I}$. Let $(N, F) \in \mathcal{I}$, and let $L\big(U_\partial(N, F)\big) = (M, G)$. Let $u \in N$. Then $\left[V(u; F_u)\right]_\partial \subseteq U_\partial(N, F)$, (where $F_u = \emptyset$ if $\Delta_1(u)$ is finite). Hence $u \in M$, and so $N \subseteq M$. It also follows that $G_u \subseteq F_u$.

We now prove $N \supseteq M$. Let $w_0 \in T^0$ and $w_0 \notin N$. We distinguish two cases.

Case (i). There exists $p_0 \in T^*_+$ with $w_0 = o(p_0)$, $t(p_0) \in \partial T$, and $t(p_0) \notin N$.

In this case, we claim first that $\left[t(p_0)\right] \notin U_\partial(N, F)$. For suppose otherwise. Then there is $u \in N$ with

$$\left[t(p_0)\right] \in \left[V(u; F_u)\right]_\partial.$$

But then $t(p_0) \in V(u; F_u) \subseteq N$, a contradiction. This establishes the claim.

Now, if $w_0 \in \partial T$, then letting $p_0 = w_0$ above we obtain $w_0 \notin U_\partial(N, F)$. Hence $w_0 \notin M$. On the other hand, if $\Delta_1(w_0)$ is finite and nonempty, the above shows that $t(p_0) \notin U_\partial(N, F)$, while $t(p_0) \in \left[V(w_0)\right]_\partial$. Hence $\left[V(w_0)\right]_\partial \not\subseteq U_\partial(N, F)$, so $w_0 \notin M$.

Case (ii). For all $p_0 \in T^*_+ \cap o^{-1}(w_0)$, if $t(p_0) \in \partial T$ then $t(p_0) \in N$.

In particular, since $w_0 \notin N$, we have that $\Delta_1(w_0)$ is finite and nonempty. Applying Definition 1.23 (iii) to w_0, we have two possibilities:

(iia) There is $w_1 \in V(w_0) \setminus \{w_0\}$ with $w_1 \in N$ and $F_{w_1} \neq \emptyset$, (and hence $w_1 \in \partial T$).

(iib) For every $w \in V(w_0) \setminus \{w_0\}$ we have $w \in N \implies F_w = \emptyset$, and there exists $w_1 \in V(w_0) \setminus \{w_0\} \setminus N$.

We will treat case (iib) first. By the assumption of case (ii), we must have $\Delta_1(w_1)$ finite. If w_1 leads to a vertex in $\partial T \setminus N$, then so does w_0, contradicting the assumption of case (ii). By the hypothesis of case (iib), w_1 must also fall under case (iib). So we obtain $w_2 \in V(w_1) \setminus \{w_1\} \setminus N$. Inductively, we obtain a sequence w_0, w_1, … in $T^0 \setminus N$ with $\Delta_1(w_j)$ finite for all j, and a path $q \in T^\infty_+ \cap o^{-1}(w_0)$ passing through all of the w_j.

In case (iia), we have $w_1 \in V(w_0) \setminus \{w_0\}$ with $w_1 \in N$ and $F_{w_1} \neq \emptyset$. Let $e_1 \in F_{w_1}$. If $t(e_1) \in N$, then Definition 1.23 (ii) (b) implies that $F_{t(e_1)} \neq \emptyset$. Hence if $e \in F_{t(e_1)}$, then $e \in F_{o(e)}$. If $t(e_1) \notin N$, then since we are in case (ii) (for w_0), $\Delta_1\big(t(e_1)\big)$ is finite. We then have case (iia) or (iib) for $t(e_1)$. If case (iib) applies,

then as before, we may extend from $t(e_1)$ an infinite directed path having infinitely many vertices not in N. If case (iia) applies, then we may extend from $t(e_1)$ a finite directed path whose last edge satisfies $e \in F_{o(e)}$, and we repeat this process.

Thus in all cases, either we have extended from w_0 an infinite directed path having infinitely many vertices not in N, or we have extended from w_0 an infinite directed path having infinitely many edges satisfying $e \in F_{o(e)}$.

Let q be this path. We claim that $[q] \notin U_\partial(N, F)$. For if it were, there would be $u \in N$ such that $[q] \in \bigl[V(u; F_u)\bigr]_\partial$. It then follows that there is a path $p \in T^* \cap o^{-1}(u)$ such that $o(p) = w_0$ and $q = pf_1f_2\cdots$ in T^∞, where $f_j \in T^1_+$ (there may be some cancellations in the product $pf_1f_2\cdots$). Since $f_1 \in \Delta_1(u) \setminus F_u$, we have $V\bigl(t(f_1)\bigr) \subseteq N$, and $F_v = \emptyset$ for all $v \in V\bigl(t(f_1)\bigr)$, by Definition 1.23 (ii) (a). Since at most finitely many f_j can have cancelled, we have contradicted the construction of q. Therefore $[q] \notin U_\partial(N, F)$, and hence $\bigl[V(w_0)\bigr]_\partial \not\subseteq U_\partial(N, F)$. Therefore $w_0 \notin M$, since $\Delta_1(w_0)$ is finite and nonempty, and we have shown that $N = M$.

Finally, we show that $G = F$. We already know that $G_u \subseteq F_u$ for all $u \in N$. Suppose that there is $u \in N$ such that $G_u \neq F_u$. Let $e_1 \in F_u \setminus G_u$. Then $t(e_1) \in N$, $G_{t(e_1)} = \emptyset$, and $F_{t(e_1)} \neq \emptyset$ (by Definition 1.23 (iii)). We choose $e_2 \in F_{t(e_1)}$. Since $e_2 \notin G_{t(e_1)}$, $t(e_2) \in N$, $G_{t(e_2)} = \emptyset$, and $F_{t(e_2)} \neq \emptyset$. Inductively, we obtain $q = e_1e_2\cdots \in T^\infty_+$ such that

$$\begin{aligned} o(q) &= u \\ G_{t(e_j)} &= \emptyset, \text{ for all } j \\ e_j &\in F_{o(e_j)}, \text{ for all } j. \end{aligned}$$

It is clear that $[q] \in U_\partial(N, G)$. In the same way as in the previous proof, it follows that $[q] \notin U_\partial(N, F)$. But since $(N, G) = U_\partial\Bigl(L\bigl(U_\partial(N, F)\bigr)\Bigr)$, it follows from the fact that $U_\partial \circ L = \mathrm{id}_{\mathcal{J}}$ that $U_\partial(N, G) = U_\partial(N, F)$. This contradiction finishes the proof. ■

We next show that a closed subset of the boundary of a directed tree corresponds to an algebra as described in Definition 1.16, corresponding to a certain subforest of the tree. Again, since we will need to apply this idea to graphs later, we define it now for directed graphs.

Definition 1.29. Let E be a directed graph, and let $(N, F) \in \mathcal{I}(E)$ be an invariant of E.

(i) $R(N, F) = \bigl\{u \in N \bigm| F_u \neq \emptyset\bigr\}$.

(ii) $E(N, F)$ is the subgraph of E given by

$$E(N, F)^0 = (E^0 \setminus N) \cup R(N, F)$$
$$E(N, F)^1_+ = E^1_+ \cap o^{-1}\bigl(E(N, F)^0\bigr) \cap t^{-1}\bigl(E(N, F)^0\bigr).$$

(iii) $S(N, F) = R(N, F) \cup \bigl\{u \in E^0 \setminus N \bigm| 0 < \#\Delta_1(u) < \infty\bigr\}$.

Proposition 1.30. *Let T be a directed tree and let $(N, F) \in \mathcal{I}(T)$. Then*

$$C_0\bigl(\partial T\bigr)/C_0\bigl(U_\partial(N, F)\bigr) \cong A\bigl(T(N, F), S(N, F)\bigr).$$

Proof. We define open subsets of $A(T)^\frown$ by

$$U(N, F) = \bigcup_{u \in N} \bigl[V(u; F_u)\bigr]$$
$$P(N, F) = \bigcup_{u \in N \setminus R(N, F)} \bigl[V(u)\bigr].$$

Thus $U_\partial(N,F) = U(N,F) \cap \partial T$ and $U(N,F) = P(N,F) \cup R(N,F)$.

Next we note that

$$A\big(T(N,F)\big) \cong A(T) \;/\; C_0\big(P(N,F)\big).$$

For the map

$$V_T(u) \mapsto \begin{cases} V_{T(N,F)}(u), & \text{if } u \in T(N,F)^0 \\ \emptyset, & \text{if } u \in N \setminus R(N,F) \end{cases}$$

extends to a Boolean ring homomorphism by Lemma 1.5. Then the kernel of the corresponding $*$-homomorphism of $A(T)$ onto $A\big(T(N,F)\big)$ is generated by the characteristic functions of $\big\{ \big[V_T(u)\big] \bigm| u \in N \setminus R(N,F) \big\}$. Hence the kernel equals $C_0\big(P(N,F)\big)$.

Now,

$$\begin{aligned} \frac{C_0(\partial T)}{C_0\big(U_\partial(N,F)\big)} &= \frac{A(T) \;/\; C_0\big(\Sigma(T)\big)}{C_0\big(U(N,F) \cap \partial T\big)} \\ &= \frac{A(T) \;/\; C_0\big(\Sigma(T)\big)}{C_0\big(U(N,F)\big) \;/\; C_0\big(U(N,F) \cap \Sigma(T)\big)} \\ &\cong \frac{A(T)}{C_0\big(U(N,F) \cup \Sigma(T)\big)} \\ &\cong \frac{A(T) \;/\; C_0\big(P(N,F)\big)}{C_0\big(U(N,F) \cup \Sigma(T)\big) \;/\; C_0\big(P(N,F)\big)}. \end{aligned}$$

We have already noted that the numerator is $A\big(T(N,F)\big)$. As for the denominator, we have

$$\begin{aligned} C_0\big(U(N,F) \cup \Sigma(T)\big) \;/\; C_0\big(P(N,F)\big) &= C_0\big(R(N,F) \cup \Sigma(T) \cup P(N,F)\big) \;/\; C_0\big(P(N,F)\big) \\ &\cong C_0\Big(\big(R(N,F) \cup \Sigma(T)\big) \setminus P(N,F)\Big) \\ &= C_0\Big(R(N,F) \cup \big(\Sigma(T) \setminus N\big)\Big) \\ &= C_0\Big(R(N,F) \cup \big(\Sigma(T) \cap T(N,F)^0\big)\Big) \\ &= C_0\big(S(N,F)\big). \end{aligned}$$

Hence $C_0(\partial T) \;/\; C_0\big(U_\partial(N,F)\big) \cong A\big(T(N,F), S(N,F)\big)$. ∎

2. Groupoids and C^*-algebras for Directed Graphs.

We now let E be a directed graph. The set E^* of (undirected) finite paths has two structures that we will use.

Definition 2.1. We let $G = G(E)$ denote the set E^* with the following groupoid structure:

$G^0 = E^0$

$r(p) = o(p)$, $s(p) = t(p)$, and $(e_1 \cdots e_n)^{-1} = \overline{e_n} \cdots \overline{e_1}$, for $p = e_1 \cdots e_n \in G$

$G^{(2)} = \big\{ (p,q) \in G \times G \bigm| t(p) = o(q) \big\}$

multiplication is given by concatenation, followed by

the removal of pairs of the form $(e, \overline{e})$

Definition 2.2. We let $\widetilde{E}$ denote the following graph:

$$\widetilde{E}^0 = E^*$$
$$\widetilde{E}^1 = \big\{(p,q) \in E^* \times E^* \;\big|\; (p^{-1},q) \in G^{(2)} \text{ and } p^{-1}q \in E^1\big\}$$
$$\widetilde{E}^1_+ = \big\{(p,q) \in \widetilde{E}^1 \;\big|\; p^{-1}q \in E^1_+\big\}$$
$$o(p,q) = p,\ t(p,q) = q,\ \overline{(p,q)} = (q,p)$$

Definition 2.3. The map $\lambda : \widetilde{E}^{**} \to E^0$ is defined by

$$\lambda(p) = o(p), \quad p \in \widetilde{E}^0 = E^*,$$
$$\lambda(\mu) = \lambda(o(\mu)), \quad \mu \in \widetilde{E}^n \text{ for } 1 \le n \le \infty.$$

Lemma 2.4. *With λ as fiber map, $\widetilde{E}$ is a bundle of directed trees.*

Proof. It is easy to check that $p,\, q \in \widetilde{E}^0$ are connected by a path if and only if $\lambda(p) = \lambda(q)$, and that in this case the path connecting them is unique. ∎

For $x \in \widetilde{E}^\infty_+/\sim$ we let $\lambda(x)$ denote the image under λ of any representative of x. The groupoid G acts on $\widetilde{E}$ in the obvious way: if $\alpha \in G$ and $p \in \widetilde{E}^0$ satisfy $s(\alpha) = \lambda(p)$ then $\alpha \cdot p$ is obtained by multiplication in G. If $\alpha \in G$ and $(p,q) \in \widetilde{E}^1$ satisfy $s(\alpha) = \lambda(p,q)$ then $\alpha \cdot (p,q) = (\alpha \cdot p, \alpha \cdot q)$. It is clear that the action of G on $\widetilde{E}$ preserves the direction.

Definition 2.5. Let E be a directed graph. We define $i : I(E) \to I(\widetilde{E})$ as follows. For $(N,F) \in I(E)$, let $i(N,F) = (\widetilde{N}, \widetilde{F})$, where

$$\widetilde{N} = t^{-1}(N)$$
$$\widetilde{F}_p = \big\{(p,pe) \;\big|\; e \in F_{t(p)}\big\}.$$

Remark 2.6. The map i is clearly injective and order-preserving.

Lemma 2.7. *Let E be a directed graph, and let $(\Lambda,\Phi) \in I(\widetilde{E})$. The following are equivalent:*
(i) $(\Lambda,\Phi) \in i\big(I(E)\big)$.
(ii) $U_\partial(\Lambda,\Phi)$ *is* $G(E)$*-invariant.*
(iii) $\Lambda = t^{-1}\big(t(\Lambda)\big)$, *and for* $p \in \Lambda$, $\Phi_p = p\Phi_{t(p)}$.

Proof. $\big((\mathrm{i}) \Rightarrow (\mathrm{ii})\big)$: Let $(\Lambda,\Phi) = (\widetilde{N}, \widetilde{F})$ for some $(N,F) \in I(E)$. Note that

$$\widetilde{F}_p = p\big\{\big(t(p), e\big) \;\big|\; e \in F_{t(p)}\big\},$$

(using the action of $G(E)$ on $\widetilde{E}$), and hence that

$$\big[V_{\widetilde{E}}(p; \widetilde{F}_p)\big]_\partial = p\big[V_{\widetilde{E}}(t(p); \widetilde{F}_{t(p)})\big]_\partial.$$

Since $\widetilde{N}$ is $G(E)$-invariant, it follows that $U_\partial(\widetilde{N}, \widetilde{F})$ is $G(E)$-invariant.

$\big((\mathrm{ii}) \Rightarrow (\mathrm{iii})\big)$: Let $p \in \Lambda$. Then

$$\big[V_{\widetilde{E}}(p; \Phi_p)\big]_\partial \subseteq U_\partial(\Lambda,\Phi).$$

If $t(q) = t(p)$, then by $G(E)$-invariance we have

$$\left[V_{\widetilde{E}}(q; qp^{-1}\Phi_p)\right]_\partial \subseteq U_\partial(\Lambda, \Phi).$$

By Theorem 1.28 we have
(1) $q \in \Lambda$,
(2) $qp^{-1}\Phi_p \supseteq \Phi_q$.
It follows from (1) that $\Lambda = t^{-1}\big(t(\Lambda)\big)$. It follows from (2) and symmetry that $p^{-1}\Phi_p = q^{-1}\Phi_q$. In particular, $p^{-1}\Phi_p = \Phi_{t(p)}$.

$\big(\text{(iii)} \Rightarrow \text{(i)}\big)$: Let $N = t(\Lambda)$. Then $\Lambda = t^{-1}\big(t(\Lambda)\big) = t^{-1}(N) = \widetilde{N}$. Since vertices of E are paths of length zero, if $p \in \Lambda$ then $t\big(t(p)\big) = t(p) \in \Lambda$. Thus $t(p) \in t^{-1}\big(t(\Lambda)\big) = \Lambda$. Hence $N \subseteq \Lambda$. For $u \in N$ let

$$F_u = \big\{e \in \Delta_{1,E}(u) \bigm| (u, e) \in \Phi_u\big\}.$$

The conditions that $(N, F) \in I(E)$ follow from those that $(\Lambda, \Phi) \in I(\widetilde{E})$. By hypothesis, then, if $t(p) = u$ we have

$$\begin{aligned}\Phi_p &= p\Phi_u \\ &= \big\{(p, pe) \bigm| e \in F_u\big\} \\ &= \widetilde{F}_p.\end{aligned}$$

Thus $(\Lambda, \Phi) = (\widetilde{N}, \widetilde{F}) = i(N, F)$. ■

Theorem 2.8. *Let E be a directed graph. The map $U_\partial \circ i$ is an order-preserving one-to-one correspondence between $I(E)$ and the collection of open $G(E)$-invariant subsets of $\partial\widetilde{E}$.*

Proof. This follows from Theorem 1.28 and Lemma 2.7. ■

We now prepare for the definition of the groupoids of a directed graph.

Definition 2.9. Let E be a directed graph, and let $x \in A(\widetilde{E})^\frown$. We define a function $\underline{\cdot} : A(\widetilde{E})^\frown \to E^{**}$ such that $o(\underline{x}) = \lambda(x)$ as follows. If $x \in \widetilde{E}^0$, then $x \in E^*$ and we let $\underline{x} = x$. If $x \in \widetilde{E}^\infty_+/\sim$, then there is a unique $\omega \in \widetilde{E}^\infty$ with $o(\omega) = \lambda(x)$ and such that some tail of ω is in (the equivalence class) x. Then we may write ω as

$$\omega = (p_0, p_1)(p_1, p_2)\cdots,$$

where $p_0 = \lambda(x)$ and for each i, $\ell(p_i) = i$. We define $\underline{x} = (p_0^{-1}p_1)(p_1^{-1}p_2)\cdots \in E^\infty$.

Lemma 2.10. *The map*

$$\underline{\cdot} : A(\widetilde{E})^\frown \to E^{**}$$

is one-to-one and has range $\big\{p \in E^{**} \bigm|$ some tail of p is directed$\big\}$. For $(\alpha, x) \in G * A(\widetilde{E})^\frown$, $\underline{\alpha x} = \alpha\underline{x}$.

Proof. The map is a bijection from $\widetilde{E}^0 \to E^*$. For $x \in \widetilde{E}^\infty_+/\sim$, if $\underline{x} = f_1 f_2\cdots$, then we may construct an infinite path in the equivalence class x by letting $p_i = f_1 f_2\cdots f_i$ (and $p_0 = o(f_1)$), and setting $\omega = (p_0, p_1)(p_1, p_2)\cdots$. This defines a left inverse for the map $\underline{\cdot}$, proving it is one-to-one. The description of the range is clearly true. To check the equivariance of the G-action, let $(\alpha, x) \in G * A(\widetilde{E})^\frown$. It is clear that

equivariance holds if $x \in \widetilde{E}^0$. So let $x \in \widetilde{E}^\infty_+/\sim$, and let $\omega = (p_0, p_1)(p_1, p_2)\cdots$ be as in Definition 2.9. Let $\alpha = e_1 \cdots e_n$, and suppose that the product $\alpha\underline{x}$ involves k cancellations. Then

$$\begin{aligned} \alpha\underline{x} &= (e_1 \cdots e_n) \cdot \big(p_1(p_1^{-1}p_2)\cdots\big) \\ &= e_1 \cdots e_{n-k}(p_k^{-1}p_{k+1})\cdots. \end{aligned}$$

But the unique element of $\widetilde{E}^\infty$ with origin equal to $\lambda(\alpha x)$ and with a tail representing αx is

$$\Big(\big(o(e_1), e_1\big)\big(e_1, e_1e_2\big)\cdots\big(e_1\cdots e_{n-k}, e_1\cdots e_{n-k}(p_k^{-1}p_{k+1})\big)\cdots\Big)$$

Then it is clear that $\underline{\alpha x} = \alpha\underline{x}$. ∎

Definition 2.11. For $S \subseteq \Sigma(E)$ we let $\widetilde{S} = t^{-1}(S) = \{p \in \widetilde{E}^0 \mid t(p) \in S\}$.

Remark 2.12. If $S \subseteq \Sigma(E)$, then $\widetilde{S}$ is open and G-invariant.

Let $S \subseteq \Sigma(E)$. We define

$$\begin{aligned} A(\widetilde{E}, \widetilde{S}) &= \bigoplus_{v \in E^0} A\big(\lambda^{-1}(v), \widetilde{S} \cap \lambda^{-1}(v)\big) \\ \partial(\widetilde{E}, \widetilde{S}) &= A(\widetilde{E}, \widetilde{S})^\wedge \; = \bigcup_{v \in E^0} \partial\big(\lambda^{-1}(v), \widetilde{S} \cap \lambda^{-1}(v)\big) \end{aligned}$$

where the union is given the inductive topology.

Definition 2.13. Let E be a directed graph. Let $S \subseteq \Sigma(E)$. The transformation groupoid $G * \partial(\widetilde{E}, \widetilde{S})$ (with the relative product topology) is called an *extended Toeplitz graph groupoid* of E (determined by the choice of S).

We recall the groupoid structure of $G * \partial(\widetilde{E}, \widetilde{S})$:

$$\begin{aligned} \big(G * \partial(\widetilde{E}, \widetilde{S})\big)^0 &= \partial(\widetilde{E}, \widetilde{S}) \\ s(\alpha, x) &= x \\ r(\alpha, x) &= \alpha \cdot x \\ (\alpha, \beta x)(\beta, x) &= (\alpha\beta, x) \\ (\alpha, x)^{-1} &= (\alpha^{-1}, \alpha x). \end{aligned}$$

Since G is discrete it is clear that s and r are local homeomorphisms, and hence that $G * \partial(\widetilde{E}, \widetilde{S})$ is r-discrete. We call the C^*-algebra of $G * \partial(\widetilde{E}, \widetilde{S})$ an *extended Toeplitz graph algebra*.

Lemma 2.14. *Let $S \subseteq \Sigma(E)$, and let $D = G * \partial(\widetilde{E}, \widetilde{S})$. The nondegenerate $*$-representations of $C_c(D)$ are in one-to-one correspondence with the pairs (π, U), where π is a nondegenerate representation of $C_c\big(\partial(\widetilde{E}, \widetilde{S})\big)$ on a Hilbert space H, and $U : G \to L(H)$ is a representation of G by partial isometries such that*

$$\begin{aligned} U_\alpha^* U_\alpha &= \pi\big(\chi_{\lambda^{-1}(s(\alpha))}\big), \\ U_\alpha U_\alpha^* &= \pi\big(\chi_{\lambda^{-1}(r(\alpha))}\big), \\ U_\alpha \pi\big(\chi_{V(p)}\big) U_\alpha^* &= \pi\big(\chi_{V(\alpha p)}\big), \end{aligned}$$

where $V(\alpha p) = \emptyset$ if $s(\alpha) \neq \lambda(p)$, and π also denotes the extension of π to the multiplier algebra of $C_0\big(\partial(\widetilde{E}, \widetilde{S})\big)$.

Proof. If $\sigma : C_c(D) \to L(H)$ is a nondegenerate $*$-representation, then $\sigma\Big|_{C_c(D^0)}$ extends to a nondegenerate $*$-representation $\pi : C_0\big(\partial(\widetilde{E}, \widetilde{S})\big) \to L(H)$. Since $\lambda^{-1}(u)$ is a clopen subset of $\partial(\widetilde{E}, \widetilde{S})$ for $u \in E^0$, the extension of π to $C_b\big(\partial(\widetilde{E}, \widetilde{S})\big)$ can be applied to $\chi_{\lambda^{-1}(u)}$.

For $\alpha \in G$,

$$\big\{\sigma(\chi_{\{\alpha\}\times V}) \;\big|\; V \subseteq \lambda^{-1}\big(s(\alpha)\big) \text{ is compact-open}\big\}$$

is a coherent net of partial isometries, hence converges in the strong operator topology to a partial isometry $U_\alpha \in L(H)$. It is clear that $U_\alpha^* U_\alpha$ is orthogonal to $\pi(\chi_{\lambda^{-1}(u)})$ if $u \neq s(\alpha)$. By the nondegeneracy of π, it follows that $U_\alpha^* U_\alpha = \pi(\chi_{\lambda^{-1}\left(s(\alpha)\right)})$. Similarly, $U_\alpha U_\alpha^* = \pi(\chi_{\lambda^{-1}\left(r(\alpha)\right)})$. The remaining properties of the $\{U_\alpha\}$ follow from the definition.

Conversely, if (π, U) are given, we define $\sigma : C_c(D) \to L(H)$ by

$$\sigma\big(\chi_{\{\alpha\}\times\lambda^{-1}\left(s(\alpha)\right)} \cdot f\big) = U_\alpha \pi(f)$$

whenever $f \in C_c(D^0)$ and supp $(f) \subseteq \lambda^{-1}\big(s(\alpha)\big)$. Then σ is a linear map, and it is easy to verify that σ is multiplicative and $*$-preserving by checking on elements of the form $\chi_{\{\alpha\}\times V}$ for $V \subseteq \lambda^{-1}\big(s(\alpha)\big)$ compact-open. ∎

Corollary 2.15. *Every $*$-representation of $C_c\big(G * \partial(\widetilde{E}, \widetilde{S})\big)$ is bounded.* ∎

(This is true for any r-discrete groupoid — see **[19]**, Proposition 3.2.)

To obtain more familiar objects we will define natural transversals in the extended Toeplitz graph groupoids. Let $S \subseteq \Sigma(E)$. We define $X(E, S) \subseteq \partial(\widetilde{E}, \widetilde{S})$ by

$$X(E, S) = \big\{x \in \partial(\widetilde{E}, \widetilde{S}) \;\big|\; \underline{x} \in E_+^{**}\big\}.$$

(Thus $X(E, S)$ is the set of points in the spectrum of $A(\widetilde{E}, \widetilde{S})$ that are represented by directed paths in E.) In the special case $S = \Sigma(E)$ we let $X(E) = X(E, \Sigma(E))$, the set of all points of $\partial\widetilde{E}$ represented by directed paths in E.

Lemma 2.16. *$X(E, S)$ is a clopen transversal in $G * \partial(\widetilde{E}, \widetilde{S})$.*

Proof. Let $x \in X(E, S)$. Then $\lambda(x) \in \widetilde{E}^0$, so we may consider the open set $[V_{\widetilde{E}}(\lambda(x))]$ in $A(\widetilde{E})\,\widehat{}\,$. Since $x \in X(E, S)$, $x \in [V_{\widetilde{E}}(\lambda(x))]$. Moreover, for any $y \in [V_{\widetilde{E}}(\lambda(x))]$, y consists entirely of directed edges of $\widetilde{E}$, and hence $\underline{y} \in E_+^{**}$. Thus $[V_{\widetilde{E}}(\lambda(x))] \cap \partial(\widetilde{E}, \widetilde{S}) \subseteq X(E, S)$, and so $X(E, S)$ is open.

Now suppose that $x \in \partial(\widetilde{E}, \widetilde{S}) \setminus X(E, S)$. Letting $\underline{x} = e_1 e_2 \cdots$, then $e_i \notin E_+^1$ for some i. Let $p = e_1 \cdots e_i \in \widetilde{E}^0$. Then $x \in [V(p)]$ and $[V(p)] \cap X(E, S) = \emptyset$. Thus the complement of $X(E, S)$ is open.

To see that $X(E, S)$ is a transversal, let $x \in \partial(\widetilde{E}, \widetilde{S})$. Let $\underline{x} = e_1 e_2 \cdots$. Then there is n with $e_i \in E_+^1$ for $i \geq n$. Let $\alpha = e_1 \cdots e_{n-1}$. Then (α^{-1}, x) has source x and range in $X(E, S)$. ∎

Definition 2.17. Let E be a directed graph. The *Toeplitz graph groupoids* are the restrictions of the extended Toeplitz graph groupoids to the transversals $X(E, S)$, and the *Toeplitz graph algebras* are their C^*-algebras:

$$\mathcal{TG}(E,S) = G(E) * \partial(\widetilde{E},\widetilde{S})\Big|X(E,S)$$
$$\mathcal{TO}(E,S) = C^*\big(\mathcal{TG}(E,S)\big),$$

for $S \subseteq \Sigma(E)$. (We let $\mathcal{TO}(E) \equiv \mathcal{TO}(E,\emptyset)$.)

The *graph groupoid* and *graph algebra* of E are special cases obtained when $S = \Sigma(E)$:

$$\mathcal{G}(E) = \mathcal{G}(E, X(E)) = G(E) * \partial\widetilde{E}\Big|X(E)$$
$$\mathcal{O}(E) = C^*\big(\mathcal{G}(E)\big).$$

Remark 2.18. In the case of a graph E with no sinks, the graph groupoid defined above is exactly the usual notion of infinite directed paths "modulo shift-tail equivalence with lag". In the case of a graph that is not row-finite, one must include the finite directed paths ending at vertices with infinite exit valence (as we have done). This idea was used in an ad-hoc way by previous authors (**[21,13,23]**).

Next we will characterize representations of $\mathcal{TO}(E,S)$ by means of generators and relations (cf. **[14,10]**).

Theorem 2.19. *The nondegenerate representations of $\mathcal{TO}(E)$ are in one-to-one correspondence with the families of operators $\{S_e\}_{e\in E^1_+}$ and $\{P_u\}_{u\in E^0}$ such that*

(i) *The $\{S_e\}$ are partial isometries and the $\{P_u\}$ are projections,*
(ii) $u \neq v \implies P_u P_v = 0$,
(iii) $\sum P_u = 1$ *in the strong operator topology,*
(iv) $S_e^* S_e = P_{t(e)}$,
(v) $P_u \geq \sum\{S_f S_f^* \mid o(f) = u\}$.

Proof. If $\pi : \mathcal{TO}(E) \to L(H)$ is nondegenerate, let

$$S_e = \pi\big(\chi_{\{e\}\times V_{\widetilde{E}}(t(e))}\big)$$
$$P_u = \pi\big(\chi_{V_{\widetilde{E}}(u)}\big).$$

It is easy to verify properties (i) - (v).

Conversely, let $\{S_e\}$, $\{P_e\}$ satisfying (i) - (v) be given. For $p = e_1 e_2 \cdots e_n \in E^*_+$, let $S_p = S_{e_1} S_{e_2} \cdots S_{e_n}$. We note the following elementary consequences of (i) - (v):

(a) $e \neq f \implies S_e^* S_f = 0$,
(b) $S_p^* S_p = P_{t(p)}$,
(c) $S_p^* S_q = 0$ if neither of p, q extends the other.

Let $\theta_p = S_p S_p^*$ for $p \in E^*_+$. We will verify the conditions of Lemma 1.5 for (the bundle of trees) $\widetilde{E}$. If $(p,q) \in \Delta_{1,\widetilde{E}}(p)$, then $q = pf$ for some $f \in E^1_+$. Then

$$\begin{aligned}\theta_q &= S_q S_q^* \\ &= S_p S_f S_f^* S_p^* \\ &\leq S_p S_p^* \\ &= \theta_p.\end{aligned}$$

Let $(p, q_1) \neq (p, q_2) \in \Delta_{1,\widetilde{E}}(p)$. Then there are $f_1 \neq f_2 \in E^1_+$, with $o(f_1) = o(f_2) = t(p)$, such that $q_i = pf_i$. Then

$$\begin{aligned} S^*_{q_1} S_{q_2} &= S^*_{f_1} S^*_p S_p S_{f_2} \\ &= S^*_{f_1} P_{t(p)} S_{f_2} \\ &= S^*_{f_1} S_{f_2} \\ &= 0. \end{aligned}$$

Thus by Lemma 1.5, there is a $*$-homomorphism $\pi_0 : C_0\big(X(E, \emptyset)\big) \to L(H)$ such that

$$\begin{aligned} \pi_0\big(\chi_{V(q)}\big) &= \theta_q,\ q \in E^*_+, \\ \pi_0\big(\chi_{V(u)}\big) &= P_u,\ u \in E^0. \end{aligned}$$

Then π_0 is nondegenerate, by (iii). Define $\pi : C_c\big(\mathcal{TG}(E)\big) \to L(H)$ as follows. The elements of $C_c\big(\mathcal{TG}(E)\big)$ can be written uniquely in the form

$$\sum \chi_{\{\alpha_1 \alpha_2^{-1}\} \times V(\alpha_2)} \cdot f_{\alpha_1, \alpha_2},$$

where the sum is taken over pairs $(\alpha_1, \alpha_2) \in E^*_+ \times E^*_+$ for which $\alpha_1 \cdot \alpha_2^{-1}$ involves no cancellations, and where $f_{\alpha_1, \alpha_2} \in C_c\big(X(E, \emptyset)\big)$ is zero except for finitely many (α_1, α_2). We define π by

$$\pi\big(\chi_{\{\alpha_1 \alpha_2^{-1}\} \times V(\alpha_2)} \cdot f\big) = S_{\alpha_1} S^*_{\alpha_2} \pi_0(f).$$

This is well-defined, since

$$\begin{aligned} \chi_{\{\alpha_1 \alpha_2^{-1}\} \times V(\alpha_2)} \cdot f = 0 &\implies \text{supp } (f) \subseteq V(\alpha_2)^c \\ &\implies S^*_{\alpha_2} \pi_0(f) = S^*_{\alpha_2} \theta_{\alpha_2} (1 - \theta_{\alpha_2}) \pi_0(f) = 0. \end{aligned}$$

π is clearly linear. It is straight-forward to verify that π is multiplicative and adjoint-preserving by checking on elements of the form $\chi_{\{\alpha_1 \alpha_2^{-1}\} \times V}$ for $V \subseteq [V(\alpha_2)]$ compact-open. ∎

Theorem 2.20. *Let $S \subseteq \Sigma(E)$. The nondegenerate representations of $\mathcal{TO}(E, S)$ are in one-to-one correspondence with the families of operators $\{S_e\}_{e \in E^1_+}$ and $\{P_u\}_{u \in E^0}$ satisfying (i) - (v) of Theorem 2.19 such that equality holds in (v) whenever $u \in S$.*

Proof. Note that equality in (v) for $u \in S$ means in particular that the sum in (v) is finitely nonzero. Let π be a nondegenerate representation of $\mathcal{TO}(E)$ corresponding to generators and relations as in Theorem 2.19. Then

$$\begin{aligned} \pi \text{ factors through } \mathcal{TO}(E, S) &\iff C_0\big(\widetilde{S} \cap X(E, S)\big) \subseteq \ker \pi \\ &\iff \big(\forall u \in S\big)\big(\exists F \subseteq \Delta_1(u) \text{ finite}\big)\Big(P_u = \sum_{e \in F} S_e S^*_e\Big) \\ &\iff \text{ equality holds in (v) whenever } u \in S. \quad ∎ \end{aligned}$$

For reference elsewhere (**[23]**), we give a slightly different form of the relations defining the algebra $\mathcal{O}(E)$. For this purpose we will let D denote the set of vertices with infinite exit valence:

$$D = \big\{u \in E^0 \;\big|\; \Delta_1(u) \text{ is infinite}\big\}.$$

Theorem 2.21. *Let E be a directed graph. The nondegenerate representations of $\mathcal{O}(E)$ are in one-to-one correspondence with the families of operators $\{S_e : e \in E^1_+\}$ and $\{P_u : u \in E^0\}$ satisfying*

(1) The $\{S_e\}$ are partial isometries and the $\{P_u\}$ are projections,
(2) $u \neq v \implies P_u P_v = 0$,
(3) $S_e^* S_e = P_{t(e)}$,
(4) $P_{o(e)} S_e = S_e$, if $o(e) \in D$,
(5) $e \neq f$ and $o(e) = o(f) \in D \implies S_e^* S_f = 0$,
(6) $u \notin D \implies P_u = \sum\{S_f S_f^* \mid o(f) = u\}$,
(7) $\sum P_u = 1$ in the strong operator topology.

Proof. This follows easily from Theorems 2.19 and 2.20. ∎

Next we will define the fundamental cocycle on the (extended) graph goupoids.

Lemma 2.22. *Let $(\alpha, y) \in G * A(\widetilde{E})\hat{}$. Then there exist unique elements $\beta_1, \beta_2 \in G$ and $x \in A(\widetilde{E})\hat{}$ such that*

(i) $(\beta_1, x),\ (\beta_2, x) \in G * A(\widetilde{E})\hat{}$
(ii) $\alpha = \beta_1 \beta_2^{-1}$
(iii) $y = \beta_2 x$
(iv) The products $\beta_1 \cdot \underline{x}$, $\beta_2 \cdot \underline{x}$, $\beta_1 \cdot \beta_2^{-1}$ involve no cancellations.

Proof. For existence, let $\alpha = e_1 \cdots e_k$ and $\underline{y} = f_1 f_2 \cdots$, and let $r \leq k$ denote the number of cancellations occurring when α and $\underline{y}$ are multiplied. We have that

$$f_i = \overline{e_{k-i+1}},\ 1 \leq i \leq r$$
$$f_{r+1} \neq \overline{e_{k-r}}.$$

Then we may put

$$\beta_1 = e_1 \cdots e_{k-r}$$
$$\beta_2 = f_1 \cdots f_r$$
$$\underline{x} = f_{r+1} f_{r+2} \cdots .$$

For uniqueness, let β_1', β_2', and x' also satisfy conditions (i)-(iv) of the statement of the Lemma. Since $\underline{y} = \beta' \underline{x}'$, there is s such that

$$\beta_2' = f_1 \cdots f_s$$
$$\underline{x}' = f_{s+1} f_{s+2} \cdots .$$

If $\beta_1' = e_1' \cdots e_\ell'$, then from $\beta_1 \cdot \underline{x} = \beta_1' \cdot \underline{x}'$ we have

$$e_1 \cdots e_{k-r} f_{r+1} f_{r+2} \cdots = e_1' \cdots e_\ell' f_{s+1} f_{s+2} \cdots .$$

Suppose $r < s$. Since paths in a tree are unique (in this case, the path from $o(e_1) = o(e_1')$ to $o(f_{s+1})$), we must have

$$e_1 \cdots e_{k-r} f_{r+1} \cdots f_s = e_1' \cdots e_\ell' = \beta_1'.$$

But then the product $\beta_1'(\beta_2')^{-1}$ involves the cancellation of $f_{r+1} \cdots f_s$, a contradiction. The assumption $r > s$ leads to a similar contradiction. Therefore $r = s$, and it follows that $\beta_1' = \beta_1$, $\beta_2 = \beta_2'$, and $x' = x$. ∎

Definition 2.23. Let $(\alpha, y) \in G * A(\widetilde{E})^\frown$. The *standard form* of (α, y) is the triple (β_1, β_2, x) satisfying (i) - (iv) of Lemma 2.22.

Definition 2.24. $c : G * A(\widetilde{E})^\frown \to \mathbf{Z}$ is defined by $c(\alpha, y) = \ell(\beta_1) - \ell(\beta_2)$, where (β_1, β_2, x) is the standard form of (α, y). For $S \subseteq \Sigma(E)$, we also let c denote the restriction to $G * \partial(\widetilde{E}, \widetilde{S})$.

Lemma 2.25. *c is a continuous homomorphism.*

Proof. The continuity is clear. Note that it follows from the definition of c that c is zero on units, and that $c(\alpha, y)^{-1} = -c(\alpha, y)$. We will prove that $c\big((\beta, y)(\alpha, y)^{-1}\big) = c(\beta, y) - c(\alpha, y)$. Let $y = g_1 g_2 \cdots$, $\beta = e_1 \cdots e_j \overline{g_k} \cdots \overline{g_1}$, $\alpha = f_1 \cdots f_p \overline{g_q} \cdots \overline{g_1}$, where $e_j \neq \overline{g_{k+1}}, g_k$, and $f_p \neq \overline{g_{q+1}}, g_q$. Then

$$\beta\alpha^{-1} = (e_1 \cdots e_j)(g_1 \cdots g_k)^{-1}(g_1 \cdots g_q)(f_1 \cdots f_p)^{-1}.$$

Suppose first that $q > k$. Then we have

$$\begin{aligned} \beta\alpha^{-1} &= e_1 \cdots e_j g_{k+1} \cdots g_q \overline{f_p} \cdots \overline{f_1} \\ &= (e_1 \cdots e_j g_{k+1} \cdots g_q) \cdot (f_1 \cdots f_p)^{-1} \\ \alpha y &= f_1 \cdots f_p g_{q+1} g_{q+2} \cdots \\ &= (f_1 \cdots f_p) \cdot (g_{q+1} g_{q+2} \cdots), \end{aligned}$$

and hence $(\beta\alpha^{-1}, \alpha y)$ has as standard form

$$(e_1 \cdots e_j g_{k+1} \cdots g_q,\ f_1 \cdots f_p,\ g_{q+1} g_{q+2} \cdots).$$

It follows that

$$\begin{aligned} c(\beta\alpha^{-1}, \alpha y) &= j + (q - k) - p \\ &= (j - k) - (p - q) \\ &= c(\beta, y) - c(\alpha, y). \end{aligned}$$

Next, if $q < k$, we may apply the above argument to $c\big((\alpha, y)(\beta, y)^{-1}\big)$. Finally, if $q = k$, we have

$$\begin{aligned} \beta\alpha^{-1} &= e_1 \cdots e_j \overline{f_p} \cdots \overline{f_1} \\ \alpha y &= f_1 \cdots f_p g_{k+1} g_{k+2} \cdots. \end{aligned}$$

Let m be the number of cancellations in the product $(e_1 \cdots e_j) \cdot (\overline{f_p} \cdots \overline{f_1})$. Then

$$\begin{aligned} c(\beta\alpha^{-1}, \alpha y) &= (j - m) - (p - m) \\ &= j - p \\ &= (j - k) - (p - q) \\ &= c(\beta, y) - c(\alpha, y). \quad \blacksquare \end{aligned}$$

Remark 2.26. Note that $c^{-1}(0)$ is an equivalence relation. For if (β_1, β_2, x) is the standard form of the element $(\beta_1\beta_2^{-1}, \beta_2 x) \in c^{-1}(0)$, then

$$\begin{aligned} r(\beta_1\beta_2^{-1}, \beta_2 x) = s(\beta_1\beta_2^{-1}, \beta_2 x) &\iff \beta_1 x = \beta_2 x \\ &\iff \beta_1 = \beta_2,\ (\text{since } \ell(\beta_1) = \ell(\beta_2)), \\ &\iff (\beta_1\beta_2^{-1}, \beta_2 x) = \big(\lambda(x), x\big) \in c^{-1}(0)^0. \quad \blacksquare \end{aligned}$$

Proposition 2.27. *Let $K \subseteq A(\widetilde{E})\widehat{\ }$ be a closed G-invariant subset, and let $c : G * K \to \mathbf{Z}$ denote the restriction of the canonical cocycle to $G * K$. Then $c^{-1}(0)$ is an AF equivalence relation.*

Proof. For $F \subseteq E$ a finite subgraph, and $n \in \mathbf{N}$, let

$$H(F,n) = \big\{(\alpha, y) \in c^{-1}(0) \mid \beta_1, \beta_2 \in F^*,\ \ell(\beta_1) = \ell(\beta_2) \le n,\ \underline{x} \in E_+^{**},$$
$$\text{(where } (\beta_1, \beta_2, x) \text{ is the standard form of } (\alpha, y))\big\}.$$

Then $H(F,n)$ is a compact-open subequivalence relation of $c^{-1}(0)$. (Compactness follows from the requirement that $\underline{x}$ consist of *directed* edges.) If $F \subseteq F'$ and $n \le n'$, then $H(F,n) \subseteq H(F', n')$, and

$$\bigcup_{F,n} H(F,n) = c^{-1}(0).$$

We will show that $H(F,n)$ is an elementary groupoid (in the sense of **[21]** III.1.1). Since $c^{-1}(0)^0$ is totally disconnected, it will follow that $c^{-1}(0)$ is an AF groupoid. (In **[21]** an AF groupoid is defined to be the inductive limit of a *sequence* of elementary groupoids. However, since each $H(F,n)$ has totally disconnected unit space, $C^*\big(H(F,n)\big)$ is an AF-algebra. Thus $C^*\big(c^{-1}(0)\big)$ is AF.)

For $y \in H(F,n)^0$, let $M(F,n,y)$ denote the set of ordered pairs $(\beta_1, \beta_2) \in F^* \times F^*$ such that

- $\exists x \in K$ with (β_1, β_2, x) equal to the standard form of an element of $H(F,n)$,
- $y = \beta_2 x$.

It is clear that $M(F,n,y)$ is finite. We claim that there is a (compact-open) neighborhood U of y in $H(F,n)^0$ such that for $y' \in U$, $M(F,n,y') = M(F,n,y)$. To see this, let $\underline{y} = f_1 f_2 \cdots$. If $\ell(\underline{y}) > n$, let $U = [V(f_1 \cdots f_{n+1})] \cap H(F,n)^0$. Then for any $y' \in U$, $\underline{y'} = f_1 \cdots f_{n+1} \cdots$, and the conclusion follows. If $\ell(y) \le n$, then $\underline{y} \in \widetilde{E}^0$. Let $R = \Delta_{1,\widetilde{F}}(y)$. This is a finite subset of $\Delta_{1,\widetilde{F}}(y)$ since F is a finite graph. We may now take $U = [V(y; R)] \cap H(F,n)$. Then for any $y' \in U$, we must have $y' = f_1' \cdots$, where $f_i' = f_i$ for $i \le \ell(y)$, and $f'_{\ell(y)+1} \notin F^1$. The conclusion now follows because $M(F,n,y) \subseteq F^* \times F^*$.

Now let $y \in H(F,n)^0$ and U be as above. For $(\beta_1, \beta_2) \in M(F,n,y)$, let $U(\beta_1, \beta_2) = \{\beta_1 \beta_2^{-1}\} \times U$. Note that for any $(\alpha, z) \in H(F,n)$ with $z \in U$, there exists $(\beta_1, \beta_2) \in M(F,n,y)$ with $\alpha = \beta_1 \beta_2^{-1}$, and thus such that $(\alpha, z) \in U(\beta_1, \beta_2)$. Further, we note that for any $(\beta_1, \beta_2) \in M(F,n,y)$,

- $U(\beta_1, \beta_2)$ is a compact-open G-set in $H(F,n)$,
- $s\big(U(\beta_1, \beta_2)\big) = U$,
- For $(\beta_1, \beta_2), (\beta_1', \beta_2') \in M(F,n,y)$ distinct, $r\big(U(\beta_1, \beta_2)\big) \cap r\big(U(\beta_1', \beta_2')\big) = \emptyset$.

It follows that

$$H(F,n)\Big|_{\bigcup\{r(U(\beta_1,\beta_2)) \mid (\beta_1,\beta_2) \in M(F,n,y)\}} \cong M(F,n,y)^2 \times U,$$

and that $H(F,n)$ is the disjoint union of finitely many such restrictions. Thus $H(F,n)$ is an elementary groupoid. ∎

Corollary 2.28. *$C^*(G * K)$ is nuclear, and coincides with $C_r^*(G * K)$.*

Proof. Let $D = G * K$. We obtain a circle action on $C^*(D)$ in the usual way:

$$\alpha_z(f)(\zeta) = z^{c(\zeta)} f(\zeta),\ \ f \in C_c(D).$$

(This extends to an automorphism of $C^*(D)$ by Corollary 2.15.) Letting $H = c^{-1}(0)$, the inclusion $C_c(H) \subseteq C_c(D) \subseteq C^*(D)$ extends to an injective $*$-homomorphism $C^*(H) \subseteq C^*(D)$ (injectivity follows easily since $C^*(H)$ is an AF algebra). By approximating from within $C_c(H)$, it is easy to see that $C^*(H) = C^*(D)^\alpha$, the fixed-point algebra. Since $C^*(H)$ is nuclear, it follows that $C^*(D)$ is nuclear (see **[18]** for a more general result in the context of coactions).

It is easy to check that the above formula for α_z defines a circle action on $C^*_r(D)$. Since $C^*(H) = C^*_r(H)$, it follows that $C^*(H) \subseteq C^*_r(D)$. Hence if E is the conditional expectation of $C^*(D)$ onto $C^*(H)$, and λ is the canonical map of $C^*(D)$ onto $C^*_r(D)$, then $E \circ \lambda = E$. It now follows that the canonical map is injective. (Alternatively, one may appeal to **[2]** 6.2.14.ii and 6.1.7.) ■

We now require two simple facts about inclusions of r-discrete groupoids.

Lemma 2.29. *(i) Let K be an r-discrete groupoid and let H be an open subgroupoid. Then $C_c(H) \subseteq C_c(K)$ extends to an injective $*$-homomorphism $C^*_r(H) \to C^*_r(K)$.*

(ii) Let $\pi : X \to Y$ be a continuous proper surjection of locally compact Hausdorff spaces. Let G be a discrete groupoid acting equivariantly on X and Y. Then $\pi^* : C_c(Y) \to C_c(X)$ extends to an injective $*$-homomorphism $C^*_r(G * Y) \to C^*_r(G * X)$.

Proof. (i) Let $u \in G^0$, and let σ_u denote the regular representation of $C_c(G * X)$ induced from the point mass at u. Then σ_u acts on $\ell^2(G_u)$ by the formula

$$\sigma_u(f)\xi(\alpha) = \sum_{\beta \in G^{r(\alpha)}} f(\beta)\xi(\beta^{-1}\alpha),\ f \in C_c(G),\ \xi \in \ell^2(G_u).$$

We have a partition of G_u into sets invariant for left multiplication by H:

$$\left\{G_u^{G^0 \setminus H^0}\right\} \cup H \backslash G_u^{H^0}.$$

Choosing a cross-section $F \subseteq G_u^{H^0}$ for $H \backslash G_u^{H^0}$, we have

$$\sigma_u \Big| C_c(H) \cong 0 \oplus \bigoplus_{x \in F} \tau_{r(x)},$$

where τ_v denotes the regular representation of $C_c(H)$ induced from the point mass at $v \in H^0$. Thus for $f \in C_c(H)$,

$$\begin{aligned} \|f\|_{C^*_r(G)} &= \sup_{u \in G^0} \|\sigma_u(f)\| \\ &\leq \sup_{v \in H^0} \|\tau_v(f)\| \\ &= \|f\|_{C^*_r(H)}. \end{aligned}$$

But if $v \in H^0$, then τ_v is a subrepresentation of $\sigma_v \Big| C_c(H)$. Hence $\|f\|_{C^*_r(G)} = \|f\|_{C^*_r(H)}$.

(ii) Let $\rho : X \to G^0$ and $\lambda : Y \to G^0$ be the fiber maps. Let $y \in Y$, let $x \in \pi^{-1}(y)$, and let σ (respectively $\widetilde{\sigma}$) denote the regular representation of $C_c(G * Y)$ (respectively $C_c(G * X)$) induced from the point mass at y (respectively x). Then σ (respectively $\widetilde{\sigma}$) acts on $\ell^2(G_{\lambda(y)})$ (respectively $\ell^2(G_{\rho(x)})$). However $\rho(x) =$

$\lambda\pi(x) = \lambda(y)$, and it is easy to see that $\widetilde{\sigma} \circ \pi^* = \sigma$. Thus for $f \in C_c(G * Y)$,

$$\begin{aligned}\|f\|_{C_r^*(G*Y)} &= \sup_{y\in Y} \|\sigma_y(f)\| \\ &= \sup_{y\in Y} \|\widetilde{\sigma}_x \circ \pi^*(f)\|, \text{ any } x \in \pi^{-1}(y), \\ &= \sup_{x\in X} \|\widetilde{\sigma}_x \circ \pi^*(f)\| \\ &= \|\pi^*(f)\|_{C_r^*(G*X)}. \qquad \blacksquare\end{aligned}$$

Corollary 2.30. *Let E be a directed graph, and let $S \subseteq \Sigma(E)$. Then $C^*\big(G * \partial(\widetilde{E}, \widetilde{S})\big)$ and $\mathcal{TO}(E,S)$ are strongly Morita equivalent.*

Proof. $\mathcal{TG}(E,S)$ is an open subgroupoid of $G * \partial(\widetilde{E}, \widetilde{S})$. By Lemma 2.29 (i),

$$C_r^*\big(\mathcal{TG}(E,S)\big) \subseteq C_r^*\big(G * \partial(\widetilde{E}, \widetilde{S})\big).$$

But by Corollary 2.28, the full and reduced C^*-algebras coincide for these groupoids. Therefore

$$\mathcal{TO}(E,S) \subseteq C^*\big(G * \partial(\widetilde{E}, \widetilde{S})\big).$$

Let $p = \chi_{X(E,S)} \in M\Big(C^*\big(G * \partial(\widetilde{E}, \widetilde{S})\big)\Big)$. Since

$$p\, C_c\big(G * \partial(\widetilde{E}, \widetilde{S})\big) p = C_c\big(\mathcal{TG}(E,S)\big),$$

we have by continuity

$$p\, C^*\big(G * \partial(\widetilde{E}, \widetilde{S})\big) p = \mathcal{TO}(E,S).$$

Moreover, p is a full projection since $X(E,S)$ is a transversal in $G * \partial(\widetilde{E}, \widetilde{S})$. $\blacksquare$

Remark 2.31. If the graph E is countable, then all groupoids under discussion are second countable. In this case the previous result follows immediately from [**17**]. Our aim, in proving Corollary 2.30 as we have done, was not only to establish the result for arbitrary graphs, but also to show that its proof is more elementary than would be implied by the use of [**17**].

Lemma 2.32. *Let E be a directed graph and let $S \subseteq \Sigma$. Then $C^*\big(G * (\Sigma \setminus S)\big)$ is isomorphic to $\bigoplus\big\{\mathcal{K}(H_u) \bigm| u \in \Sigma \setminus S\big\}$, where $H_u = \ell^2\big(E^* \cap t^{-1}(u)\big)$.*

Proof. Since $\Sigma \setminus S$ is a discrete set, we have that $G * (\Sigma \setminus S)$ is the disjoint union of the (discrete) transitive groupoids $G * \big(E^* \cap t^{-1}(u)\big)$, $u \in \Sigma \setminus S$. The lemma now follows from the fact that the C^*-algebra of a discrete transitive groupoid is elementary. $\blacksquare$

Corollary 2.33. *Let E be a directed graph and let $S \subseteq \Sigma$. There is an exact sequence*

$$0 \longrightarrow I_S \longrightarrow \mathcal{TO}(E,S) \longrightarrow \mathcal{O}(E) \longrightarrow 0,$$

where $I_S = \bigoplus\big\{\mathcal{K}_u \bigm| u \in \Sigma \setminus S\big\}$, and $\mathcal{K}_u$ is isomorphic to the algebra of compact operators on a Hilbert space of dimension $\#\big(E_+^* \cap t^{-1}(u)\big)$.

Proof. This follows from Corollary 2.30, Lemma 2.32 and [**21**], Proposition II.4.5.(i). $\blacksquare$

Now let $E_1 \subseteq E_2$ be an inclusion of directed graphs. We have an inclusion of bundles of directed trees $\widetilde{E}_1 \subseteq \widetilde{E}_2$, hence an injective $*$-homomorphism $A(\widetilde{E}_1) \to A(\widetilde{E}_2)$ (Remark 1.19). Let $S_2 \subseteq \Sigma(E_2)$. We define $S_1 \subseteq \Sigma(E_1)$ by

$$(*) \qquad S_1 = \{ v \in E_1^0 \cap S_2 \mid \Delta_{1,E_2}(v) \subseteq E_+^1 \}.$$

Then $\widetilde{S_1}$ is the set defined in Lemma 1.20 by the inclusion $\widetilde{E}_1 \subseteq \widetilde{E}_2$ and the set $\widetilde{S}_2 \subseteq \Sigma(\widetilde{E}_2)$. Thus the following diagram commutes, has surjective columns and injective rows:

$$\begin{array}{ccc} A(\widetilde{E}_1) & \longrightarrow & A(\widetilde{E}_2) \\ \downarrow & & \downarrow \\ A(\widetilde{E}_1, \widetilde{S}_1) & \longrightarrow & A(\widetilde{E}_2, \widetilde{S}_2). \end{array}$$

From Lemma 2.29 we obtain a composition of injective $*$-homomorphisms:

$$C^*\big(G(E_1) * \partial(\widetilde{E}_1, \widetilde{S}_1)\big) \to C^*\big(G(E_1) * \partial(\widetilde{E}_2, \widetilde{S}_2)\big) \to C^*\big(G(E_2) * \partial(\widetilde{E}_2, \widetilde{S}_2)\big).$$

We obtain a corresponding injective $*$-homomorphism of Toeplitz graph algebras, which we state as a theorem.

Theorem 2.34. *Let $E_1 \subseteq E_2$ be an inclusion of directed graphs, and let $S_2 \subseteq \Sigma(E_2)$. Let $S_1 \subseteq \Sigma(E_1)$ be defined as in $(*)$. Then there is an injective $*$-homorphism*

$$\mathcal{TO}(E_1, S_1) \to \mathcal{TO}(E_2, S_2). \qquad \blacksquare$$

Theorem 2.35. *Let E be a directed graph. Let $\mathcal{W}$ be any collection of subgraphs which is directed by inclusion and for which*

$$\bigcup_{F \in \mathcal{W}} F^0 = E^0$$

$$\bigcup_{F \in \mathcal{W}} F^1 = E^1.$$

For $S \subseteq \Sigma(E)$, and $F \in \mathcal{W}$, let $S_F \subseteq \Sigma(F)$ be as above. Then

$$\mathcal{TO}(E, S) = \varinjlim_{\mathcal{W}} \mathcal{TO}(F, S_F).$$

Proof. The coherence of the system of $*$-homomorphisms follows from Remark 1.21. Note that $\varinjlim_{\mathcal{W}} A(\widetilde{F}, \widetilde{S}_F)$ is dense in $A(\widetilde{E}, \widetilde{S})$ by the Stone-Weierstrass theorem, since each $B \in \mathcal{A}(\widetilde{E})$ is the image of a set in $\mathcal{A}(\widetilde{F})$ for all large enough F. Since $G(E) = \bigcup_{\mathcal{W}} G(F)$, the equality stated in the theorem holds. $\blacksquare$

3. Structure of Graph C^*-algebras.

The results in this section are parallel with similar results in other work (e.g. **[9,13,14]**). We show here that they follow easily in our general framework.

Definition 3.1. Let E be a directed graph. A *cycle* in E is a path $\alpha = e_1 \cdots e_n \in E_+^*$ such that $o(\alpha) = t(\alpha)$ and $o(e_i) \neq o(e_j)$ for $1 \leq i < j \leq n$. An *exit* of α is an edge in $\Delta_1\big(o(e_i)\big) \setminus \{e_i\}$ for some i. The cycle α is *terminal* if α has no exit. The cycle α is *transitory* if it is not terminal, and if for each exit f of α we have $o(e_j) \notin V\big(t(f)\big)$ for all j; (in other words, α has an exit, but no exit from α leads back to α).

Lemma 3.2. *Let E be a directed graph. Let $u \in E^0$, and suppose that E has a non-terminal cycle (whose vertices are contained) in $V(u)$. Then the restriction of $G(E) * \partial\widetilde{E}$ to $\left[V_{\widetilde{E}}(u)\right]_\partial$ contains an open G-set Z with $s(\overline{Z}) \subseteq \left[V_{\widetilde{E}}(u)\right]_\partial$ and $r(\overline{Z}) \subsetneqq s(Z)$.*

Proof. Let $\alpha = e_1 \cdots e_n$ be a non-terminal cycle with vertices in $V(u)$. We may assume that $o(\alpha)$ emits an exit from α. Let $U = \left[V_{\widetilde{E}}\big(o(\alpha)\big)\right]_\partial$ and let $Z = \{\alpha\} * U$. Then Z is the required a G-set. (Any directed path that begins with an exit from α represents a point of U that is not in the range of Z.) ∎

Theorem 3.3. *Let E be a directed graph. Then $\mathcal{O}(E)$ is an AF algebra if and only if E has no cycles.*

Proof. Suppose first that E has a cycle α. If α is terminal, we may let N be the set of vertices in α and take F to be trivial. Then $(N, F) \in \mathcal{I}(E)$, and the corresponding ideal of $\mathcal{O}(E)$ is isomorphic to a matrix algebra over the continuous functions on the circle. Since $\mathcal{O}(E)$ has a non-AF ideal, $\mathcal{O}(E)$ is not AF. If α is not terminal, then let Z be as in Lemma 3.2. Then χ_Z is a partial isometry in $\mathcal{O}(E)$ whose final projection is a proper subprojection of its initial projection. It follows that $\mathcal{O}(E)$ is not AF.

Now suppose that E has no cycles. Then if F is any finite subgraph of E, F^* is a finite set, and hence $\mathcal{TO}(F, S_F)$ is finite dimensional. Then Theorem 2.35 implies that $\mathcal{O}(E)$ is AF. ∎

We recall from **[1]** that an r-discrete groupoid is called *locally contractive* if for every nonempty open subset U of the unit space there is an open G-set Z with $s(\overline{Z}) \subseteq U$ and $r(\overline{Z}) \subsetneqq s(Z)$ (see also **[16]**).

Theorem 3.4. *Let E be a directed graph. Then $G(E) * \partial\widetilde{E}$ is locally contractive if and only if there are no terminal cycles, and $V(u)$ contains a cycle for every $u \in E^0$.*

Proof. We first suppose that E has no terminal cycles and that $V(u)$ contains a cycle for every $u \in E^0$. Then Theorem 3.2 applies to every $u \in E^0$. Since every nonempty open subset of $\partial\widetilde{E}$ contains $V(u)$ for some $u \in E^0$, local contractivity follows from Theorem 3.2.

We now prove the converse. If α is a terminal cycle, then $[\alpha^\infty]$ is an isolated point of $\partial\widetilde{E}$, and hence is an open set with no contracting subset. If $u \in E^0$ is such that $V(u)$ does not contain a cycle, consider the subgraph D of E with $D^0 = V(u)$ and $D^1 = o^{-1}(D^0) \cap t^{-1}(D^0)$. By Theorem 3.3, $\mathcal{O}(D)$ is AF. Therefore the restriction of $G(E) * \partial\widetilde{E}$ to (the open set) $\left[V_{\widetilde{E}}(u)\right]_\partial$ does not contain a G-set as in the definition of local contractivity. Hence $G(E) * \partial\widetilde{E}$ is not locally contractive. ∎

Definition 3.5. Let E be a directed graph. We call E *cofinal* if the following two properties hold.

(i) For every $u \in E^0$ and for every $p = e_1 e_2 \cdots \in E_+^\infty$, there is $\alpha \in E_+^*$ with $o(\alpha) = u$ and $t(\alpha) = o(e_j)$ for some j.

(ii) $E^0 \setminus \Sigma(E) \subseteq V(u)$ for all $u \in E^0$.

Remark 3.6. The first condition is the notion of cofinality (see e.g. **[13]**) used in the case of row-finite graphs. The second condition is necessary when there are vertices with infinite exit valence.

Theorem 3.7. *Let E be a directed graph. Then $G(E) * \partial\widetilde{E}$ is minimal if and only if E is cofinal.*

Proof. Suppose first that E is cofinal. Let $U \subseteq \partial\widetilde{E}$ be a nonempty open invariant set. We will show that $U = \partial\widetilde{E}$. Let $x \in \partial\widetilde{E}$. By invariance of U we may assume that x is represented by a directed path $p \in E_+^{**}$, where $t(p) \in E^0 \setminus \Sigma(E)$ if $\ell(p) < \infty$. Again by invariance, we may find $u \in E^0$ with $\left[V_{\widetilde{E}}(u)\right]_\partial \subseteq U$. By cofinality there is $\alpha \in E_+^*$ such that $o(\alpha) = u$, and $t(\alpha)$ is one of the vertices of p. But then $\alpha^{-1}x \in U$, so that $x \in U$ by invariance.

Now suppose that $G(E) * \partial\widetilde{E}$ is minimal. Let $u \in E^0$. Put $N_0 = V(u)$. Having defined N_i for $i < i_0$, let

$$N_{i_0} = \bigl\{v \in \Sigma(E) \bigm| t\bigl(\Delta_1(v)\bigr) \subseteq \bigcup_{i<i_0} N_i\bigr\}.$$

Let $N = \cup_i N_i$ and let $F_v = \emptyset$ for $v \in N$. Refering to Definition 1.23, all conditions are evidently true, so that $(N, F) \in \mathcal{I}(E)$. Since $N \neq \emptyset$, minimality and Theorem 2.8 imply that $N = E^0$. Since $N \setminus V(u) \subseteq \Sigma(E)$, condition (ii) of cofinality holds.

It is easily shown by induction that if $v \in N$ and if $e_1 e_2 \cdots \in E_+^\infty$ with $o(e_1) = v$, then $t(e_n) \in V(u)$ for all n large enough. Now let $p \in E_+^\infty$. By minimality and Theorem 2.8 there exist $v \in N$, and $\alpha \in E^*$ with $t(\alpha) = v$, such that $[p] \in \alpha\bigl[V_{\widetilde{E}}(v)\bigr]_\partial$. Hence there is $q = e_1 e_2 \cdots \in E_+^\infty$ with $o(q) = v$ such that $p = \alpha q$. Since $t(e_n) \in V(u)$ for large n, condition (i) of cofinality holds. ■

We recall that an r-discrete groupoid is called *essentially free* if the set of units having trivial isotropy is dense in the space of units.

Theorem 3.8. *Let E be a directed graph. Then $G(E) * \partial\widetilde{E}$ is essentially free if and only if E has no terminal cycles.*

Before proving Theorem 3.8 we need some lemmas. We will call a *circuit* any directed path of positive length whose origin and terminus coincide.

Lemma 3.9. *Let E be a directed graph and let $x \in \partial\widetilde{E}$. Then x has nontrivial isotropy if and only if $x = [\beta\gamma^\infty]$ for some circuit γ.*

Proof. If x has the indicated form, then $\beta\gamma\beta^{-1}$ fixes x. Conversely, suppose $\alpha x = x$ with $\ell(\alpha) > 0$. Then necessarily $\underline{x}$ has infinite length. Let $(\alpha, \underline{x}) \in G(E) * \partial\widetilde{E}$ have standard form (β_1, β_2, y). If $\ell(\beta_1) = \ell(\beta_2)$, then since $\beta_1 y = x = \beta_2 y$ we must have $\beta_1 = \beta_2$, contradicting the assumption on $\alpha = \beta_1\beta_2^{-1}$. So we may assume $\ell(\beta_1) < \ell(\beta_2)$. But then from $\beta_1 y = \beta_2 y$ we find that $\beta_2 = \beta_1\gamma$. Then $y = \gamma y$, so $y = \gamma^\infty$. It follows that $x = \beta_1\gamma^\infty$. ■

Lemma 3.10. *Let E be a directed graph and let $\alpha \in E_+^*$ be a circuit. Then $\alpha = \beta\gamma\delta$ where β, γ, $\delta \in E_+^*$ and γ is a cycle.*

Proof. Let γ be the portion of α between two consecutive occurrences of a repeated vertex in α. ■

Lemma 3.11. *Let E be a directed graph having no terminal cycles. Then for every $u \in E^0$ there is $p \in E_+^{**}$ such that $o(p) = u$ and $[p] \in \partial\widetilde{E}$ has trivial isotropy.*

Proof. Suppose first that $V(u)$ contains a non-transitory cycle. Then this cycle contains a vertex which emits an edge that is not in the cycle, but leads back to a vertex in the cycle. Thus there are α, β, γ, $\delta \in E_+^*$ such that

(i) $\beta\gamma$ is the chosen cycle,

(ii) $o(\delta) = o(\beta$, $t(\delta) = t(\beta)$, and the first edge of δ is not the first edge of β,

(iii) $o(\alpha) = u$ and $t(\alpha) = o(\beta)$.

Then the path $p = \alpha\beta\gamma\delta\gamma(\beta\gamma)^2\delta\gamma(\beta\gamma)^3 \cdots$ has the required properties.

Now suppose that u leads only to transitory cycles. In this case there exists a path $p \in E_+^{**}$ with $o(p) = u$, and $t(p) \in E^0 \setminus \Sigma(E)$ if $\ell(p) < \infty$, such that p contains no cycles. Such a path can be constructed inductively by exiting any cycle encountered. By Lemma 3.10, such a path will contain no circuits. By Lemma 3.9, it will have the required properties. ■

Proof of Theorem 3.8. Suppose first that E has no terminal cycles. If $x \in \partial\widetilde{E}$ has nontrivial isotropy, Lemma 3.9 implies that $x = [\beta\gamma^\infty]$ for some circuit γ. Let $u = o(\gamma)$. Let p be as in Lemma 3.11. Then $x_n = [\beta\gamma^n p]$ are points in $\partial\widetilde{E}$ with trivial isotropy that converge to x.

Conversely, suppose that E has a terminal cycle, α. Then $[\alpha^\infty]$ is an isolated point in $\partial\widetilde{E}$, and has non-trivial isotropy. ∎

Corollary 3.12. *Let E be a directed graph, and let $S \in \Sigma(E)$. Then $G(E) * \partial(\widetilde{E}, \widetilde{S})$ is essentially free if and only if E has no terminal cycles.*

Proof. Recall $\Sigma(\widetilde{E})$ is a discrete clopen subset of $A(\widetilde{E})^\frown$. Moreover, every point of $\Sigma(\widetilde{E})$ has trivial isotropy. Therefore $\partial(\widetilde{E}, \widetilde{S}) = \partial\widetilde{E} \cup \big(\Sigma(\widetilde{E}) \setminus \widetilde{S}\big)$ has a dense set of points with trivial isotropy if and only if the same is true of $\partial\widetilde{E}$. ∎

We next recall from **[21]**, Definition II.4.3, that an r-discrete groupoid is called *essentially principal* if its restriction to each closed invariant subset of the unit space is essentially free. The characterization in graph terms rests on the following lemma.

Lemma 3.13. *Let E be a directed graph. Then there exists $(N, F) \in \mathcal{I}(E)$ such that $E(N, F)$ has a terminal cycle if and only if E has a terminal or transitory cycle.*

Proof. Suppose that E has a transitory cycle, α, and let C denote the vertices of the edges in α. Let $D = \{e \in E^1_+ \mid o(e) \in C,\ t(e) \notin C\}$. Then for every $u \in t(D)$ we have $V(u) \cap C = \emptyset$. So we may let $N = \cup_{e\in D} V\big(t(e)\big)$, and $F_u = \emptyset$ for $u \in N$. Then it is easy to see that $(N, F) \in \mathcal{I}(E)$, and α becomes a terminal cycle in $E(N, F)$.

Conversely, suppose $(N, F) \in \mathcal{I}(E)$ is such that $E(N, F)$ has a terminal cycle. Note that by Definition 1.23 (iia),

$$\bigcup_{u \in N\setminus R(N,F)} V(u) \subseteq N \setminus R(N, F),$$

and that $N \setminus R(N, F) = E^0 \setminus E(N, F)^0$. Now let α be a terminal cycle in $E(N, F)$. With C and D as before, we must have

$$\bigcup_{e\in D} V(u) \subseteq E^0 \setminus E(N, F)^0.$$

Therefore α is either terminal or transitory in E. ∎

Theorem 3.14. *Let E be a directed graph. Then $\mathcal{G}(E)$ is essentially principal if and only if E has no terminal or transitory cycles.*

Proof. By Theorem 2.8 and Proposition 1.30, the closed invariant subsets of $\partial\widetilde{E}$ are of the form

$$\partial\big(E(N, F)^\sim, S(N, F)^\sim\big),$$

for $(N, F) \in \mathcal{I}(E)$. By Corollary 3.12, the restriction to such a set is essentially free if and only if $E(N, F)$ has no terminal cycles. By Lemma 3.13, no $E(N, F)$ has a terminal cycle if and only if E has no terminal or transitory cycles. ∎

We may assemble some of the above results as follows.

Theorem 3.15. *Let E be a directed graph.*

(i) *$\mathcal{O}(E)$ is a nuclear C^*-algebra.*

(ii) *If E has no terminal or transitory cycles, then the lattice of ideals in $\mathcal{O}(E)$ is isomorphic to the lattice of invariants $\mathcal{I}(E)$.*

(iii) *$\mathcal{O}(E)$ is simple if and only if E is cofinal and has no terminal cycles.*

(iv) *$\mathcal{O}(E)$ is simple and purely infinite if and only if E is cofinal, has no terminal cycles, and if $V(u)$ contains a cycle for every $u \in E^0$.*

Proof. Statement (i) follows from Corollaries 2.28 and 2.30. Statement (ii) follows from Theorem 3.14 and **[21]**, Theorem II.4.5(iii). Statement (iii) follows from (ii) and Theorems 3.7 and 3.8. Statement (iv) follows from (ii), Theorem 3.4, and **[1]** (or **[16]**, whose argument adapts immediately to r-discrete groupoids). ∎

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